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Master's Thesis

# Dynamic Programming Approach in Continuous-Compensation, Infinite-Horizon Principal-Agent

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Tiivistelmä — Referat — Abstract

We consider a so-called principal-agent problem, where our aim is to construct an optimal contract that maximises utilities for the contractor, *the principal*, and for the effort-exerting party, *the agent*. In our setting, the time-horizon of the contract is infinite, and the agent receives a continuously paid compensation for exerting effort. Our main goal is to establish a problem introduced in Sannikov (2008) and characterise an optimal contract by restricting the menu of feasible contracts, an approach inspired by Cvitanić et al. (2018).

We begin with an extensive literature review, where we review the continuous-time principal-agent problems and further motivate the scope of this thesis. We start from the notable article Holmström et al. (1987), and we progress towards the works Williams (2008), Sannikov (2008) and Cvitanić et al. (2018).

Following the review, we construct the problem and lay the mathematical foundations for it. We focus on a new benchmark setting, adapted from Sannikov (2008) and Cvitanić et al. (2018). We first define the so-called controlled state equation, and construct the canonical probability space. Then, we introduce then impose few assumptions regarding the core concepts, and identify the problems of the agent and the principal and characterise their objective functions.

After characterising the problem, we characterise the optimal contract and show that the optimal contract maximises the principal's profit. We characterise the difference function of Sannikov (2008) to the agent's optimisation problem, and then follow Cvitanić et al. (2018) on the reduction of the problem. The reduction is done by restricting the possible menu of contracts and thus reduce the non-standard problem to a dynamic programming problem. We introduce the corresponding Hamiltonian functionals, together with the value functions to the both principal and the agent. Furthermore, we introduce a family restricted processes, which we show to characterise the optimal contract. We finish with showing that the optimal contract exists even with the notion of retirement.

Having completed the main technical contribution, that is, having solved for the optimal contract, we briefly discuss the results and their implications against previous literature. Additionally, we discuss the possible extensions to our research.



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*Joka keinussa jumalten keinuu*
*Väliä taivaan ja helvetin heiluu*
*Hän kokee huiput ja kuilut kun keinuu*
*Kun keinuu*[1]

– Eino Leino

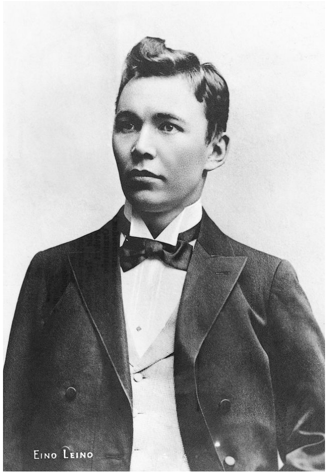


**Figure 1:** Eino Leino (Picture Source: Finnish Heritage Agency)

[1] The text is, in fact, not written by the Finnish poet Eino Leino (1878–1926). Instead, it is written by Finnish singer-songwriter Jare Henrik Tiihonen for his song ”Keinu” (2015). This seemingly peculiar mix-up between the two Finnish superstars was inspired by a https://twitter.com gag started by a Twitter user **@***Tero_Hoo* amidst the Eino Leino Day – the Day of Poetry and Summer – celebrations on July 6, 2020. This so-called *Twitter thread* is titled "Post your favourite Eino Leino quotation – wrong answers only!". The thread connects to the domain of principal-agent problems by incentivising the platform users to exert effort. The *original poster* can only observe the output, but not the hidden information, which is the real writer of the quotes masquerading as works of Eino Leino. This ”contract” is peculiar in a sense that it gives no profit or monetary value to the parties involved. For more information and the origin of the joke, see https://twitter.com/Tero_Hoo/status/1280050008839110657 (viewed on November 10, 2020).

# Contents

# i. Preamble

On the topic and selected research question: for my whole career as an economics student, which began after my mathematics' MSc. graduation, I have felt a distinctive pull towards theoretical economic research. The main reason for this might be my inherent lazyness: analysing and editing real-world panel data is a big hustle, and better to be saved for the possible post-graduation studies. The other reason might be the lucrativeness of theoretical economic research: how the theories are sometimes linked in policy-making and how the theories are formed on the foundation of the philosophy of science or empirical research. However, I can not shake the feeling that getting involved in a theoretical economic research made me understand a great deal of aspects from my previous degree of mathematics. No matter how deep my reasons are to be involved in the theoretical research business, the topic to this thesis was stumbled upon by accident. Inspiration to this thesis was found as a direct quote from the Introduction of Cvitanić et al. (2018), page 2: "In recent years a different continuous-time model has emerged and has been very successful in explaining contracting relationship in various settings – the infinite-horizon problem in which Principal may fire/retire Agent and the payments are paid continuously, rather than as a lump-sum payment at the terminal time, as introduced in another seminal paper, Sannikov [32]. We leave for a future paper the analysis of the Sannikov's model using our approach."

On writing: despite the thesis consisting of adademic text, I[2] have done my best to keep the language colourful and informative. The text may or may not include some poorly-timed jokes and subtle references, but despite all the joking and jest, I have kept the text precise, informative and easy to follow.

As the reader will probably notice, the text is partially filled with footnotes. Despite *Noël Coward* famously saying that "Having to read footnotes resembles having to go downstairs to answer the door while in the midst of making love."[3], the footnotes are something that I, as a writer, deeply enjoy: they allow me to make remarks without making the sentence at hand increasingly complex, and in addition, they are an excellent resource to make the thesis as mathematically exhaustive as possible: citing a theorem whilst explaining its core contents in

[2] Do note, that instead of the personal pronoun "we", this preamble is written with the personal pronoun "I" in order to better distinct the narrator's personal role in the preamble.

[3] See Grafton (1999), pages $69 - 70$.

a footnote is an extremely good practice to keep the thesis informative.

As a sidenote, I need to address the Eino Leino ”quote” presented. I strongly believe that the quote tells more about the author than it tells about economics or principal-agent problems. However, a fine thesis always reflects the mindset of its author.

I want to express my thanks to my proof-reading team (you know who you are) for pointing out thousands of errors. Thank you to the The Häme Student’s Foundation for offering me a research room, without which the thesis would not be finished. Thank you to my employer for the extreme flexibility around the time of submitting the thesis. Thank you, *PhD candidate Teemu Pekkarinen* for encouraging and supporting me in the field of microeconomics, as well for being an excellent and flexible teaching assistant in various economics courses I attended. Finally, thank you to my thesis supervisor *Professor Klaus Kultti* for his sometimes curious, yet extremely helpful supervision, which perfectly mirrored my curious way of writing this thesis.

Helsinki, November 10, 2020,
Topias Tolonen

# 1. Introduction

Dynamic incentives, and in particular, constructing optimal contracts that maximise utilities for the contractor, *the principal*, and for the effort-exerting party, *the agent*, is a central piece of interest in microeconomics. Solutions to these so-called *principal-agent problems* are widely used in applications, ranging from finance to university management. Examples include how a corporation would properly incentivise its managers, and how the Ministry of Culture and Education should give monetary incentives to the higher educational institutions in Finland, and how these institutions would react to such incentives[4]? This thesis, albeit heavily theoretical and not delving deeper into the realm of higher educational politics in Finland, extends the theoretical research of these problems, with which many such topics can be better understood and explained. We establish a problem introduced in Sannikov (2008) and characterise an optimal contract by restricting the menu of feasible contracts, an approach inspired by Cvitanić et al. (2018). Both the problem and the optimal contract are well-defined and feasible within the established *moral hazard* framework. The scope of this research provides a welcome inspection and extension to the research and literature of principal-agent problems, stochastic control and dynamic optimisation.

Principal-agent problems and such moral hazard problems were given a continuous-time generalisation by the notable pioneers Bengt Holmström and Paul Milgrom in their article Holmström et al. (1987), where they discussed a continuous-time extension for the model in discrete time $t = \{0, 1, 2, \dots\}$. Holmström and Milgrom used the theory of Brownian Motion[5] and modelled the problem in a static way, where the agent controls their output by controlling the drift rate of an account vector that has built-in random fluctuations within. The most notable contribution of Holmström et al. (1987) is that the optimal contract is shown to be linear in aggregate profits, and that this result reflects empirical research. The microeconomic interest for expanding their research is driven by their finite time horizon and exponential utility

[4]The discussion on governmental steering of higher educational institutions is ongoing, especially during the time of the publication of the Ministry's Core Funding Model for Universities 2021–2024. The microeconomic discussion of the funding model revolves around the question that whether the provided monetary incentives distort the behaviour of the universities away from their "purpose" or strategic goals? This phenomenon is thoughtfully discussed in, for example, Vartiainen et al. (2018).

[5]See, for example, Durrett (2010), chapter 7.

functions. Especially the advancements in the theory of stochastic control have made extensions to their research possible. The work of Holmström et al. (1987) was notably expanded[6] by Sannikov (2008), who expanded the problem by modelling[7] the problem as an infinite-horizon problem, during which the agent receives a constant consumption based on the past output. The economic interest in the research of Sannikov (2008) is that their research allows the framework for promotions, retirement, and different types of utilities – all of which reflect the contractual needs of the empirical world.

While the work of Holmström et al. (1987) focused on agents with exponential utility functions and risk-neutral principals, the generalisation of the model for more general utility functions is discussed in Cvitanić et al. (2009), where the authors use the theory of stochastic maximum principle and backward stochastic differential equations to solve the optimal contract in such setting. Later on, this approach was expanded in Cvitanić et al. (2016) and Cvitanić et al. (2018), where the authors considered a model where the agent is, in addition to the drift of the output process, able to control the volatility of the process[8], and the optimal contract was found by reducing the number of feasible contracts. The motivation for their approach is easily understood: by controlling the volatility, the agent has a chance to tune the risk within the contract at any given time, and reflect this on the compensation received. Additionally, the major result in Cvitanić et al. (2018) is that by reducing the menu of possible contracts, the problem can be reduced without decreasing the possible profits of the principal, and transforming the problem into a dynamic programming problem. By reducing the number of contracts, the research leaves room for more generalised utility functions, or for relaxing other assumptions. In this thesis, the goal is to formulate the optimal contract in this benchmark setting – the continuous-compensation and infinite-horizon setting of Sannikov (2008) – with the tools and theoretical approach provided by Cvitanić et al. (2018).

Our research finds answers to the following questions: do we find a matching optimal contract compared to Sannikov (2008) in a continuous-time setting, even if we follow the approach of Cvitanić et al. (2018) and restrict the number of feasible contracts? Does the optimal contract exist when we introduce the option for the agent to retire? Are we able to find an optimal solution to the problem if we relax the assumptions concerning the agent's utility function? How does our characterised optimal contract compare to the previous literature explaining the empirical and theoretical contracting relationships, mainly regarding the classic linearity result of Holmström et al. (1987)?

[6]Other notable articles that have flourished on the foundations of Holmström and Milgrom include, but are not limited to, Cvitanić et al. (2009) and Schättler et al. (1993). We explore the developments of the research in continuous-time principal-agent problems more broadly expanded in chapter 2.

[7]That is, in addition to the original properties of the model in Holmström et al. (1987), where the time horizon is continuous over a single period, $0 \leq t < 1$, and the consumption to compensate the agent's efforts were paid as a lump-sum payment at the end of the period.

[8]This extension was first explored in Sung (1995).

We argue that these questions are of relevance, and studying them more carefully paramount when answering questions regarding contracts between two parties.

First, using a more refined mathematical setting to extend the principal-agent problem of Sannikov (2008), and finding an optimal contract has a few economic points of interest. We find an optimal contract, which is characterised by the agent's dynamic value function. The economic contribution of our optimal contract is that even in the infinite-horizon and continuous-time setting, where compensation is paid and effort is exerted continuously, the optimal contract is characterised at a single observation point $t$. The option to retire the agent accounts for the so-called income effects, and together with the restricted family of feasible contracts characterise the optimal contract even when the effort-exertion would become too expensive in the long run. In addition, as pointed out in Sannikov (2008), this benchmark setting allows to study the optimal mix of short and long-term incentives, and in addition, it gives a theoretical framework when analysing employee wage fluctuations.

Secondly, from a microeconomic standpoint, it is interesting to see how the restriction of feasible contracts allows for relaxing the assumptions on utility functions. In our setting, we impose only two assumptions, *boundedness* and *invertibleness*[9], to the agent's utility function $U_A$. It is fairly straight-forward to argue that the dynamic programming approach, where the number of possible contracts is reduced, refers to a trade-off between the number of contracts and the generality of the utility functions. Despite the fact that empirical research, such as Friend et al. (1975), suggests that exponential utilities reflect the behaviour of agents, the flexibility given by our research gives the option for extending the research to more detailed agent study.

Thirdly, we discuss the relevance of the dynamic programming approach in the context of principal-agent problems. In the end of the day, the continuous-compensation model may end up being an unnecessary complicated model and producing the same results of aggregate linearity as in Holmström et al. (1987). However, as pointed out in Sannikov (2008), the optimal contract across a long time horizon is of interest: Sannikov (2008) argues that in his model, the long-term optimal contract is not linear and thus sways from the aggregate linearity result. This result is found in Holmström et al. (1987) and Schättler et al. (1993), and furthermore discussed from an empirical perspective in Friend et al. (1975).

We find an optimal contract by reducing the number of feasible contracts, and show that this optimal contract indeed has the same value for the principal, despite the fact that the contract was picked from a restricted menu. The agent has a bounded and invertible utility, while the principal's utility function is non-decreasing and concave. Both of these utilities can be represented as standard exponential utilities, but we characterise the utilities without specifying them. We show that the optimal contract can be approximately linear, and that the optimal contract is similar to the optimal contract derived in Sannikov (2008). Furthermore,

[9]For the cencors and all of other readers: we did check that this is, quite surprisingly, a real word.

we discuss the implications of the optimal contract.

To reiterate, this thesis provides an insightful extension to the problem established in Sannikov (2008). We follow the example of Cvitanić et al. (2018) and restrict the menu of possible contracts. We also review relevant mathematical tools and the construction of the continuous-time principal-agent model, where the compensation is paid continuously, the time-horizon is infinite, and the agent has an option to retire. We solve the problem by reducing the menu of contracts and then verify that the optimal contract indeed maximises the principal's profit. After solving the problem in the benchmark setting, we review the properties of an optimal contract against the relevant results in the relevant principal-agent literature. We then conclude with a discussion and point out points for further research.

## Outline of the Thesis

In chapter 2 we further motivate the scope of this thesis. and examine the advancement of the research in continuous-time principal-agent problems. we begin with the research of Holmström and Milgrom in their notable article Holmström et al. (1987), where they construct a continuous-time principal-agent problem and derive an optimal contract, where the wage is found to be linear in aggregate output. Then, we examine the immediate extensions to Holmström et al. (1987), and review how the generalisation of the problem and identifying further sufficient conditions affected the results. Afterwards, we scrutinise the article Sannikov (2008), where the continuous-time problem is extended to infinite-horizon, and where the agent is compensated continuously. In their model, the agent has a possibility of retirement. In Sannikov (2008), the optimal contract is found to be linear in short time horizon. As a final piece of the review, we approach the works Cvitanić et al. (2009), Cvitanić et al. (2016) and Cvitanić et al. (2018), and explore the notion of dynamic programming in the context of principal-agent problems.

After the literature review, in chapter 3 we construct the problem and lay the mathematical foundations for it. This chapter focuses on a new benchmark setting, adapted from Sannikov (2008) and Cvitanić et al. (2018). We focus on constructing the mathematical basis in the first section 3.1, where we define the so-called controlled state equation, and construct the canonical probability space. In the following sections 3.2 and 3.3, we introduce then impose few assumptions regarding the core concepts, such as cost function and contracts. Furthermore, we identify the problems of the agent and the principal and characterise their objective functions. Before venturing further, finalise the setting and introduce the possibility of retirement in section 3.4. We finish the chapter with discussing the game theoretical properties of the problem, along with the implications of dynamic value function. Additionally, we comment the constructed problem and its assumptions briefly on section 3.5 in the light of relevant previous literature, namely Holmström et al. (1987), Sannikov (2008) and Cvitanić et al. (2018).

Then, we move on to chapter 4, which contains the main technical contribution of the thesis. We characterise the optimal contract and show that the optimal contract maximises the principal's profit. In section 4.1, we characterise the difference function of Sannikov (2008) to the agent's optimisation problem, and then follow Cvitanić et al. (2018) on the reduction of the problem. The reduction is done by restricting the possible menu of contracts and thus reduce the non-standard problem to a dynamic programming problem. We introduce the corresponding Hamiltonian functionals, together with the value functions to the both principal and the agent. In 4.2, we introduce a family restricted processes, which we show to characterise the optimal contract. Additionally, we impose the characteristics of Sannikov (2008): the contract has no termination point, the principal has the option to retire the agent and the agent has an option to accept the retirement. We finish with showing that the optimal contract exists even with the notion of retirement.

After we characterise and validate the optimal contract, in chapter 5 we comment the results against the previous literature, and briefly discuss the implications of our research. We discuss the optimal contract against the aggregate linearity result of Holmström et al. (1987), against the results derived in Sannikov (2008). Additionally, we discuss the possible extensions to our research.

We finish the thesis with chapter 6, we summarize the takeaways from the research and conclude the thesis.

# 2. From the Aggregate Linearity Result to Dynamic Programming in Principal-Agent Problems – Reviewing The Literature

In this chapter, we review the previous literature regarding continuous-time principal agent problems[10], starting from the aggregate linearity result derived in Holmström et al. (1987). The literature review is presented in order to support some of the claims and motivational comments in this thesis. This selectively trimmed review broadly expands the topic at hand for a more comprehensive overview for the reader. We review the most important literature regarding the research of these problems, and use the microeconomic insights of the authors to motivate the path which leads to the continuous-compensation, infinite-horizon model of Sannikov (2008) and to the dynamic programming approach of Cvitanić et al. (2018).

If the reader is enthusiastic and well-informed on the state-of-art principal-agent related literature, they are welcome to skip directly to the chapter 3, where we construct our model.

## 2.1 About Continuous-time Principal-Agent Problems

This section covers the introduction to the continuous-time principal-agent problem as it was first formulated in Holmström et al. (1987). Furthermore, we review its extension in Holmström et al. (1991), Schättler et al. (1993), Sung (1995), Sung (1997), Schättler et al. (1997), Müller (1998), Müller (2000), and Hellwig et al. (2002).

[10] Originally, a sketch of this literature review was a part of an extensive research proposal handed in for my Thesis Proposal Seminar course, labelled as ECOM-361 in Master's Programme in Economics. The seminar was a delight, and it helped the author to hone the topic of this thesis from a general idea brought up in Cvitanić et al. (2018).

### 2.1.1 The Ones who Started it All – the Linearity Result of Holmström and Milgrom (1987)

The setting in the groundbreaking article Holmström et al. (1987) is fairly simple: they consider a moral hazard problem, where they assume a strictly risk-averse agent together with a risk-neutral principal. The agent is allowed to constantly alter the drift of his output process: this *effort-exertion* is unobservable by the principle. The continuity in their setting comes as an approximation of a multinomial discrete model.

For a brief moment, let us discuss the assumptions regarding the risk-aversion of the agent and risk-neutrality of the principal. For standard Moral Hazard problem, these assumptions are intuitively explainable: for example, Hölmstrom (1979) and Mirrlees (1999) argue that if both the principal and the agent are risk-neutral, then the principal can achieve a first-best solution by "selling the firm" to the agent – that is, by making the agent a residual claimant. If, on the other hand, the agent is risk-averse, the main insight in the before mentioned articles is that providing incentives to a risk-averse agent is costly.[11] However, the risk-aversion of the agent, especially with exponential utility functions, is mathematically fairly simple to deal with. In addition, for example in Friend et al. (1975) it is argued that there is empirical and experimental basis on the risk-aversion of the agent. Finally, if the principal were risk-averse along with the agent, the analysis of the utility maximization problems would get unnecessarily complicated: after all, according to Friend et al. (1975), the general results of the models would stay the same. Thus, the risk-neutrality of the agent and the risk-aversion of the principal were well justified in Holmström et al. (1987), and continued to be so even further.

To contextualize the setting further, we briefly introduce how Holmström et al. (1987) discusses discrete time models. In their research, they consider both single period model and a so-called finitely repeated model. They conclude that the optimal contract depends on aggregates, and furthermore explain how the continuous-time approximation was founded. In the discrete time models, the optimal compensation schemes can be shown to be linear functions of more comprehensive aggregates. The key takeaway is that the continuous-time Brownian Motion serves as an approximation to any multinomial model that satisfies the following three key assumptions:

- The agent acts $T$ times in a period, say, numerated to a single year.
- The multinomial model has a large number of periods, e.g., "$T$ is large" (sic, Holmström

[11] There are multiple reasonings behind this, which line up with the expected behaviour of "real world" agents: incentives for the agent require that the compensation is based on the effort. This implies, that the participation is costly. Furthermore, this effect troubling in the scenario where the effort is unobservable, and thus the principal provides incentives based on a stochastic signal. In addition, if the incentive costs rise too high, the principal might be better off by relaxing the incentives and letting the agent choose lower output. The implications of this are discussed later in the sense of the optimal contracts.

et al. (1987)).

- Costs and profits, in each period, are relatively small compared to the agent's and the principal's risk tolerances.
- Number of outcomes that the principal distinguishes in a single period is smaller than the dimension of the agent's control set.

The resulting optimal contract is linear in the end-of-the-period levels of the different diffusions of the output processes.

The linearity can be vaguely explained by imposing few assumptions, and building upon them.[12] If the agent's utility is additively separable in consumption and effort, and is unbounded below in consumption, then there is no optimal solution, since first-best solution can be approximated arbitrarily close using a compensation scheme of the following sort. Consider that the agent receives fixed wage until very low output and very low wage for very low output. Holmström et al. (1987) argues that this scheme is feasible since the normal distribution in the process has the property where the low outputs are very much more likely when the agent shirks than when he does not.[13]

In addition, to construct a scheme, the principal requires high information about the agent's beliefs and preferences, in addition to the technologies she controls. Their model, with the two-wage scheme mentioned above and where the compensation is paid as a function of profits over time, leads the agent to work hard only when it happens necessary to avoid a disaster. This is intuitively a linear scheme, since it applies the same fixed pressure on the agent no matter how the past performance or pressure were and thus leads to more appropriate choice of effort over time. As a main contribution, Holmström et al. (1987) models this intuitive train of thought by Brownian Motion where the agent can control the drift of the process. When assuming that agent has exponential utility function[14] and that the cost of drift control is monetary, the optimal incentive is linear in aggregate output.

That is, the main result of Holmström et al. (1987) is that when assuming a single-period, constant effort output in a second-best scenario[15] and when the agent's utility is strictly risk-averse and exponential, then the optimal contract is linear in the aggregate output.

[12] As always in the field of economics: the assumptions are the key. The assumptions in the Principal-Agent model are examined further in this review.

[13] This "scheme" is referred to a two-wage setting, which characterizes the model.

[14] Notice that the assumption was that the agent is strictly risk-averse – together with the exponential utility function this is usually characterized as a CARA utility. Another thing to pay attention to is that the CARA utility abstracts from income effects, where higher utility agents demand more compensation for the same effort level.

[15] By second-best scenario we refer to the scenario where the actual effort level is not observed, rather only the aggregated output.

### 2.1.2 Expansions to Holmström and Milgrom (1987) – Further Sufficient Conditions and Generalised Framework

Article Schättler et al. (1993) discusses the continuous-time model of Holmström et al. (1987) further. The goal of their research is to provide a large-scale theoretical foundation for the Principal-Agent problem, of which the model of Holmström et al. (1987) is a special case. As one of the main results in their paper, the they give sufficient conditions for the first-order approach to the continuous-time principal-agent problems, which they claim are easily verified.

A major addition to Holmström et al. (1987) is that Schättler et al. (1993) notices that the principal-agent problem can be related to a stochastic optimal control problem by relaxing the incentive compatibility constraint of the agent to the first-order conditions. These conditions take the form of semi-martingale representation for the agent's salary[16].

In their model, the semi-martingale representation describes the salary as the aggregate of four different components. The components are the opportunity cost for the agent, compensation for the agent for controlling the drift of the process, compensation error due to the fact that the salary is not based on the actual effort and instead on the principal's observation, and finally a risk premium for the compensation error. Schättler et al. (1993) interprets that these four components are consistent with the model of Holmström et al. (1987), and they further note that the exponential utility is not required in this model – the analogous results hold even if the preferences where additively separable. In their model, they also replace the principle's salary function by another semi-martingale representation, and then the corresponding salary functions are implemented by substituting the control into the semi-martingale representation.

Furthermore, Schättler et al. (1993) argues in a fantastic way that the principal needs to focus only on those representations corresponding to a specific controls which the agent actually uses to maximize their utility. This causes the limitation to principle's possible actions, which in turn makes the class of salary functions implementing a given control much smaller. This same notion is notably used as inspiration for Sannikov (2008). In addition, Schättler et al. (1993) claims that the reasoning for linearity result in Holmström et al. (1987) is "mainly intuitive", and that it results mostly to a weak stochastic solution for the problem. They fix this by proving a sound and rigorous theoretical framework for the model.

To take matters further, Sung (1995) directly expands the framework of Schättler et al. (1993) by allowing the agent to increase the diffusion rate – *the volatility* – either publicly or privately. As we have learned, the outcome process retains a Brownian Motion with an alterable drift and volatility. However, in this paper both the principal and the agent have CARA[17] preferences, and as in Holmström et al. (1987), the principal can not observe the

[16]Theoretical framework for the martingale representation in the stochastic control problem can be for example found in Rishel (1970).

[17]CARA stands for Constant Absolute Risk-Aversion, and this is often modelled as $u(c) = 1 - e^{-\alpha c}$, for some

mean, only the output process, but she may or may not observe the variance and thus knows the riskiness of the contract. The main result for Sung (1995) is that the optimal contract is still linear in the output sense, even when the agent is able to control the variance.

As an fascinating and extremely important remark, Sung (1995) relaxes a key assumption of Holmström et al. (1987) and still derives the main result which is the linearity of the optimal contract. This result of course serves to be a further inspiration to develop the model of Holmström et al. (1987), as the relaxation of the assumption implies that the Brownian Motion model is a robust in some sense. We argue that this was one of the key reasons for Cvitanić et al. (2009) to even further develop the goal, together with the inspiration attained from Sannikov (2008). Of course, as Sung (1995) argued, allowing the agent to control the volatility of the process constraints the principal's problem, and furthermore, that due to the stationarity assumption in the cost function and CARA preferences, the agent's optimal responses to a given contract inherit the stationarity of the cost function. This argument makes the optimal contract similar to those in Holmström et al. (1987) with only one extra time-state independent constraint. This results in a stationary observations and thus stationary decision-making – which equals the setting in Holmström et al. (1987)! Furthermore, the result implies the linearity in profits.

If the principal is risk-neutral, as she is in Holmström et al. (1987), Sung (1995) argues that the sufficient condition for the optimal contract when the principal can or can not observe the diffusion rate is that the agent's cost function is convex, and that it is additively separable in mean and variance. In other words, the agent is able to control only the "noise" in the process without affecting the drift. This furthermore aligns with the risk-neutral preferences of the principal, since the decreased noise implies the improvement in the agent's incentives.

Furthermore, Sung (1995) also applies the model and the setting to showcase moral hazard problem and incentives in an applied managerial setting.

Sung (1997) mathematically formalizes the framework in Schättler et al. (1993) and Sung (1995) by examining various accidents and insurance companies: Holmström et al. (1987) examined the properties where low-output realizations become more probable when the agent shirks. Sung (1997) ties this argument of normality and formalizes the setting in a more applied setting. However, the formalization of Sung (1997) was minor compared to the extensive advancement of both Schättler et al. (1993) and Sung (1995).

In the article Müller (1998), the authors explore the possibilities to extend the models in Holmström et al. (1987) and its generalised version by Schättler et al. (1993). Main addition to the previous literature is that Müller (1998) notices that when the principal's problem is separable[18], and when CARA utilities are assumed for both the principle and the agent, the

constant $\alpha$.

[18]This comes from the relaxation argument in Schättler et al. (1993).

optimal contract is linear even when the agent's control is observed and enforced at no cost[19].

On the other hand, Müller (2000) tweaks the previous models by enforcing a rule that the control revisions can only be taken place in a discrete time. They continue to show that no matter how close these revision times are to continuous-time, the linear contracts are not optimal as the principal can approach the first-best solution asymptotically with a random spot check and a suitably chosen set of revision functions on the side. Müller (2000) argues that this "expansion" to the model is widely used in practice, since the optimal solution is not restricted to the first contract made. The solution can be approximated even if the revisions are applied to the outputs separately. However, it is also noted that in practice, this approach can be deemed expensive if the audits at each revision point are costly and if the number of the revisions surge.

As a final extension of Holmström et al. (1987) reviewed in this section, Hellwig et al. (2002) examines further conditions on when discrete-time models converge into the solution of Holmström et al. (1987), and study the further structure underlying the linearity that was proposed in it. In addition, it is proposed that further clarification was needed to separate the linearity in profits, which was the aggregate result in Holmström et al. (1987), and the "linearity in accounts". By linearity in accounts, Hellwig et al. (2002) means that they assume CARA utilities for the agent and stationarity of technology in the process. Based on these assumptions, the principal observes the outcome paths of the agent's output, and the incentive problem studied has a stationary solution with an incentive scheme which is a linear function of the frequencies. With this solution, the different "one-period outcomes" are observed. Furthermore, this implies that the incentives pressure is only reliant on the future expectation. As opposed to the Holmström et al. (1987), Hellwig et al. (2002) studies this relationship more deeply.

In addition, Hellwig et al. (2002) notices that the linearity of profits require more assumptions than the beforementioned CARA utilities and the stationarity in the technology. In order to reach the linear optimal contract, Holmström et al. (1987) imposed an additional assumption of the principle only the time path of aggregate profits[20]. Hellwig et al. (2002) notices that this assumption has no discrete-time analogy. The main result of Hellwig et al. (2002) was then to establish the multinomial multi-period model in discrete time, where the linearity of profits is meaningfully translated from the continuous-time model.

Apart from the literature we reviewed above, the setting of Holmström et al. (1987) is approached by numerous other articles: Bolton et al. (2001) approaches the setting from a game-strategic perspective, Ou-Yang (2003) explores the notion of principal-agent in a world of financial portfolio management, and Detemple et al. (2001) tackles more general utility

[19]The authors call this the "first-best sharing rule", as opposed to the "second-best" scenario, where the control can not be observed and instead only the aggregated output is.

[20]This means, that between the different periods observed in Holmström et al. (1987), the principal does not see the profits of any single-period: only the total.

functions.

## 2.2 Continuous Compensation and Dynamic Programming Approach

This section covers the major reference articles Sannikov (2008), Williams (2008), Cvitanić et al. (2009) and Cvitanić et al. (2018), and introduces their settings and methods. There are major similarities between these articles, but as always, the devil is in the details. The end goal of Cvitanić et al. (2018) is to provide a sound mathematical framework to allow the generalisation of the assumptions used by changing the problem into a dynamic programming problem[21]. On the other hand, the major accomplishment of Sannikov (2008) was to provide a simple framework in an infinite-horizon setting, where the agent receives a continuous compensation and has an option to either retire, quit or get an promotion. we focus discussion on these two articles, since the goal of the thesis is to explore the model of Sannikov (2008) further with the mathematical ideas given by Cvitanić et al. (2018).

### 2.2.1 A Different Approach – Continuous Compensation with a Chance of Retirement

The motivation for the research in Sannikov (2008) is to examine the factors which make the optimal contract prices increase or decrease over time, and to examine to which degree current and future events motivate the agent. Note that the latter was also the motivating issue behind Holmström et al. (1987). Furthermore, Sannikov (2008) discusses a chance for retirement, and more specifically the conditions under which the agent eventually reaches retirement in the optimal contract. The goal in their research is to analyse the cost of incentives and the dynamic properties of the optimal contract depend on the contractual environment.

As in many other economic models, Sannikov (2008) makes a few assumptions: there is a risk-averse agent[22] and a risk-neutral principal, that are tied together forever after the employment or contract starts. The contract which the agent is taking upon specifies the agent's consumption at every given moment based on the past output. Compared to Holmström et al. (1987), they also consider a so-called income effect, where it becomes costlier to compensate agent with high income for the same effort.[23] In addition, the agent's utility is assumed to be

[21]This is a similar trick as was done in Schättler et al. (1993), where the assumptions were relaxed by restricting the feasible menu of contracts.

[22]Note that the assumption in Holmström et al. (1987) was a strictly risk-averse agent.

[23]When the income effect is taken into consideration, the compensation of the agent becomes costly in the long run, then it would be optimal for the principal to allow agent to exert zero effort. It is also possible to suspend the agent for a short period of time, but under the assumptions of Sannikov (2008), these contracts

bounded from below, which matches the Holmström et al. (1987) and its "two-wage" argument.

It is crucial to notice that Holmström et al. (1987) limited the model into a single-period setting, where the end of the period was usually normalized to 1. It is paramount to understand that in Sannikov (2008), the time horizon is set to infinite, and that the consumption is paid continuously until a retirement or a firing or a promotion is reached at some $t$, $0 < t < \infty$.

Sannikov (2008) examines how the optimal contracts are changed if the agent may quit, be replaced or promoted. It is also noticed that while the agent may not control his volatility of the process, a higher effort requires higher volatility of the agent's value since the agent has more incentives to push more effort when his value depends more on the output. Thus the agent's drift always trends in the direction where it is cheaper to provide incentives — also when the agent is patient, that is, the incentive provision is cost-free and the discount factor nears 0, his continuation value[24] has no drift.

Over short-time intervals[25], Holmström et al. (1987) notoriously showed that the optimal contracts are linear in aggregate output when assuming an exponential utility function with a monetary cost of effort. These utilities had no income effect. Furthermore, Holmström et al. (1991) discussed that the model of Holmström et al. (1987) is exceptionally well-suited for compensation over short time period[26]. Sannikov (2008) discussed that in their model, the short-time interval optimal contracts are also approximately linear, which lines up with the discussion in Holmström et al. (1987). Major differences[27] arise when the infinite-horizon scheme actually pans out, and the time draws out further. Sannikov (2008) also argues that the discrete models, without the framework of the stochastic differential equations, "produce a more limited set of results". The discrete time models are expanded by Spear et al. (2005) as formulating them as recursive models, but Sannikov (2008) finds that the restrictions applied to solve these recursive models are not very informative about the optimal paths of wages in the contracts.

However, it the setting of Sannikov (2008), the long run optimal contract involves some complex non-linear patterns of the agent's wage and effort. When the contract binds the agent forever, which is indeed the setting, the agent eventually reaches retirement when their continuation value reaches low end-point or high end-point, the agent receives constant stream of consumption and stops their effort. Spear et al. (2005) shows that retirement happens when the agent's continuation payoff becomes very low or very high. In Sannikov (2008) it is, as stated

would be sub-optimal.

[24]His expected future payoff.

[25]Remember that Holmström et al. (1987) discussed that the continuous-time model is an approximation of a multinomial model over a single period with many simultaneous periods, or that agent acts multiple times during a period.

[26]This attribute was shortly discussed in Schättler et al. (1993), Sung (1995) and Hellwig et al. (2002).

[27]This is very likely to be tied to the income effect we discussed earlier, as is suggested in Hellwig et al. (2002).

before, assumed that the agent's consumption utility is bounded from below, which means that when the agent reaches this point, the payments to the agent must stop. Alternatively, when the consumption peaks too high, due to the income effects assumed the compensation becomes too costly. It is also argued that it would be sensible to avoid the permanent retirement by-consumption by allowing the agent to suspend her effort temporarily. However, Sannikov (2008) goes on to show that if the agent is equally patient as the principal, in the optimal contract the agent eventually reaches the permanent retirement.

In their benchmark setting, the agent can also quit if she was forced to stop consumption at a low-retirement point, assuming that she has acceptable or relevant outside opportunities. The agent may or may not be replaceable. In the case she is replaceable, the principle is able to hire a new agent upon the retirement of the old agent. Of course, a high-valued retirement point can be replaced with a promotion, which allows the agent higher human capital and higher expected effort output. Sannikov (2008) finds that higher upfront payments cause the agent's continuous value to drift up, away from quitting. This is heavily tied to the fact that better outside options make the agent harder to retain. In addition, the chance of promotion motivates the agent with her future improvements to the quality of life.

As opposed to the usual practical world where the majority of the contracts are heavily focused in short-term incentives, Sannikov (2008) uses a model where the ratio of the volatilities of her consumption and continuation value measures the mix of short-term and long-term incentives: the main results are that the optimal contract is focused on the short-term incentives when the agent has good outside options, which are to interfere with her long-run incentives. Furthermore, from the principal's point of view, when the principal has a greater flexibility in promoting or replacing the agent, the long-term incentives are focused in the optimal contract. This leads to the result where the agent puts higher effort and the principal gets higher profit when the optimal contract is relied on stronger long-term incentives.

The work of Sannikov (2008) was partly inspired by Phelan et al. (1991), in which a long-term discrete time way to compute optimal contracts was discussed. There were many similarities in the methods of these two papers, but due to their computation relying in linear programming, together with multiple iterations to test the convergence, the method of Sannikov (2008) with differential equations are far less demanding.[28]

A notion of so-called patient agent is also discussed in Sannikov (2008). In such case the the agent's discount rate reaches 0. It was shown in Fudenberg et al. (1990), that efficiency becomes attainable, when the agent is patient. However, Sannikov (2008) shows that with patient agent, the dynamics lose a large portion of relevancy – the agent's continuation value becomes drift-less and the effort is decreasing in agent's value.

A similar continuous-time model is also explored in Williams (2008), where the partial

[28]It might be sensible to explore the possibilities of Phelan et al. (1991) with the current computation power available, but we'll leave it for another time.

differential equations and forward-backward stochastic differential equations are used to model the optimal contract with recursive state variables, but due to greater generality of the setting, the optimal contract is not discussed in detail.

## 2.3 Stochastic Maximum Principle and the Limitation of Feasible Contracts – Dynamic Programming Approach

The articles discussed before mainly used CARA utility functions or risk-neutrality for the principle, which in Holmström et al. (1987) and further articles resulted in the optimal contract being linear in the aggregate output. Cvitanić et al. (2009) explores this setting further by discussing more general utility functions by using the stochastic maximum principle and forward-backward stochastic differential equations to characterize the optimal contracts. Williams (2008), on the other hand explores a more general and dynamic setting while still using the stochastic maximum principle.[29]

The article Williams (2008) builds upon the foundation of Holmström et al. (1987) and Schättler et al. (1993), and the main contribution to these previous articles is that in while the Holmström et al. (1987) considers only a single transfer of wealth from the principle to the agent, the model in Williams (2008) allows transfers to occur continuously throughout the contract period. Now, the agent's utility process becomes another state variable, which makes the dynamic setting history-dependant.[30]. This approach is also similar to the approach in Sannikov (2008), and their results share similarities. However, it is stated in Williams (2008) that while the simpler setup of Sannikov (2008) results in a more complete characterization of the optimal contract, their setting covers much more general models with natural dynamics, and unfortunately does not allow for a complete characterization of the contract. The article also explores the concept of so-called hidden-states, where the agent could have an access to a hidden part of consumption, which is hidden from the principle.

One of the major differences to previous literature, which Williams (2008) failed to clarify, was that the Stochastic Maximum Principle of Bismut (1973) was applied to characterize the optimal conditions for the agent facing a specified contract. Williams (2008) also did not discuss the nature of the second-best scenario, which is something Cvitanić et al. (2009) returns to discuss. A second-best scenario was described in Müller (1998), where the agent's effort is unobservable by the principle and there is only one-time payoff at the end of the contract.

[29]Dynamic Programming approach is also – interestingly enough – used in Elie et al. (2017), where the approach is distinctly similar to Cvitanić et al. (2009) and Cvitanić et al. (2016), but instead of a single agent, they use infinitely many Agents, and are interested in the convergence of the optimal contract in a finite time.

[30]This attribute was also discussed in Spear et al. (2005).

The main result for Cvitanić et al. (2009) in this sense is that the optimal contract for non-exponential preferences is a *non-linear function of aggregated output*, as opposed to the linear case in the other settings. In addition, Cvitanić et al. (2009) uses the stochastic maximum principle to give the necessary conditions for the problem. They cover for the loss of exponential utility and further assume that the cost function is quadratic in agent's effort and that that the utility is additively separable. These assumptions, together with the maximum principle, lead to an optimal contract.

The model in Cvitanić et al. (2018) considers a very general principal-agent problem, where a lump-sum is paid at the end of the contracting[31], and with the stochastic maximum principle and the recent results on the second order backwards stochastic equations, the technicality goes up to retrieve the more general results.

Cvitanić et al. (2016) is mainly considered to be a precursor article for Cvitanić et al. (2018), where the tools of the latter article are used in a less general setting. They consider a model where the agent controls her drift and diffusion rate in a special case of CARA preferences for both the principal and the agent. They argue that the in addition to the fact that the optimal contract depends linearly on the aggregate output, it also depends on the risk the output has been exposed to via the risk's quadratic variation. They consider a multi-dimensional model with arbitrary utility functions with also relaxing the Markovian or stationarity assumptions made in the previous models. In their method, they reduce the availability of contracts by considering only the contracts that allow a dynamic programming representation of the agent's value function, for which they furthermore identify the effort for the agent, which is incentive compatible. Furthermore, they show that this reduction of the contract space does not lead to loss of generality.[32]. Finally, they show that the supremum of Principal's expected utility over the restricted set of contracts is equal to the supremum of expected utility over all feasible contracts.

The setting is further generalised in Cvitanić et al. (2018), where a more general utility functions are discussed. It is found that the optimal contract is a function of the terminal value only: in the first-best contract scenario, the ratio of the agent's and the principal's marginal utilities is constant, which implies that the ratio is a linear function of the principal's utility. Finally, Cvitanić et al. (2018) highlights some examples, for which their method is suitable.

Recently, the literature of principal-agents – like many other aspects of economics and social sciences – has also been advanced to the realm of computational solutions, due to the large amounts of data and computational power available. For example, in principal-agent problems

[31] In this case, the setting is resembling the original model on Holmström et al. (1987).

[32] They borrow the idea of infinitesimal decomposition structure of the agent's dynamic value function from Sannikov (2008), and then restrict the family of admissible contracts. Furthermore, for such contracts, the principal can identify the optimal policy for the agent, which is the corresponding policy which maximizes the corresponding Hamiltonian.

a numerical solution can be used to characterise an otherwise difficult optimal contract, as was done in Azar et al. (2018). In addition, in the context of dynamic programming and principal-agent problems, many optimal contracts are recently characterised using backward stochastic differential equations[33]. For many[34] backward stochastic differential equations, as pointed out in Tolonen (2017), solutions can not be found analytically, and instead to find a solutions, numerical methods have to be used[35].

We have now extensively covered the relevant literature of continuous-time principal-agent problems, focusing the research to the literature which progresses to continuous-compensation problems and dynamic programming approaches. Let us now move to combine these two aspects in the continuum of principal-agent problems by constructing the theoretical framework of our research.

[33]See, for example, in Elie et al. (2017) and Cvitanić et al. (2018).

[34]For example, if the so-called *generator* of the backwards stochastic differential equation is not linear.

[35]For example, numerical approximation of backward stochastic differential equations is covered in Ma et al. (2002).

# 3. The Benchmark Setting

Let us take a moment to construct the model and the problem formally, and then transfer the intuition between the abovementioned research settings and a formal mathematical construction into a solid foundation for our research. Principal-agent problems are of stochastic in nature: the *hidden information* aspect in what the principal can observe, combined with the random process spanned by the agent's effort, together form a wonderful nest for randomness, which requires tools considering random processes, random variables and probability spaces. Thus, it is natural for us to start with introducing the mathematical foundation to the problem, simultaneously while defining the problem itself.

## 3.1 Delve into the World of Continuous Random Variables

In this section, we briefly introduce the mathematical set-up for our setting by defining the canonical probability space with filtrations and martingales, along with the controlled state equation and the admissible control models. We motivate the mathematical approach via the output process spanned by the agent and create a basis exhaustive enough to support the optimal contract in our infinite-horizon, continuous-compensation principal-agent problem. Since this section is seemingly technical in nature, we keep the reader comfortable and explain all of the concepts used or referred to, and in addition define the notations used.

In our principal-agent problem, we have two decision-makers: *the principal* and *the agent*[36]. The agent can make an action to exert effort, through which he has a partial control over the *output process*, which is something the principal is interested in. The principal can not observe the effort exerted by the agent, but instead she sees the output, and can incentivise the agent to provide better results for both parties. We now construct a mathematical basis for the output process, and then progress to define the so-called *controlled state equation*, according to which the process evolves.

In order to meaningful define the output process as a random process, we create a proba-

[36]Later on, in order to avoid confusion, we refer to the principal as "she" and to the agent as "he" whenever necessary.

bility space in our infinite-horizon setting. The first step is to define the sample space as a set of all continuous maps from the interval $[0, \infty)$ to $\mathbb{R}^d$ for some dimension $d > 0$:

$$\Omega := C^0([0, \infty), \mathbb{R}^d). \tag{3.1}$$

The output process of the agent is then defined as a continuous random variable

$$X_t(x) = x_t \tag{3.2}$$

for all $x \in \Omega$ and $t \in [0, \infty)$[37]. In order to preserve a canonical history into the decision-making, we have to introduce a corresponding filtration as

$$\mathbb{F} := \{\mathcal{F}, t \in [0, \infty]\}, \tag{3.3}$$

where $\mathcal{F} := \sigma(X_s, s \leq t)$ is the sigma-algebra[38] generated by the observed history at $t \in [0, \infty)$. Then, we require a probability measure. With $\mathbb{P}_0$ we denote the Wiener measure[39] on the pair $(\Omega, \mathcal{F}_\infty)$, and for any $\mathbb{F}$-stopping time[40] $\tau$, with $\mathbb{P}_\tau$ we denote the regular conditional probability distribution of $\mathbb{P}_0$ with regards to $\mathcal{F}_t$. This is well-defined, since the sigma-algebra $\sigma$ is countably generated and thus[41] $\mathbb{P}_\tau$ is a proper probability distribution. Furthermore, $\mathbb{P}_0$ is independent of the realisation $x_t$, since the output is generated by a Brownian Motion, which in turn gains independent and stationary increments[42]. Finally, we denote the collection of all probability measures under the pair $(\Omega, \mathcal{F}_\infty)$ as $\text{Prob}(\Omega)$.

Additionally, all processes in this thesis are of a similar form as in Cvitanić et al. (2018): we work with processes $\phi : [0, \infty] \times \Omega \to E$, where E is some Polish space[43], and the processes are $\mathbb{F}$-optional. $\mathbb{F}$-optionality is well-defined in Cvitanić et al. (2018), but to put it simply, it means a collection of processes measurable[44] with regards to a $\sigma$-algebra generated by $\mathbb{F}$-adapted and continuous processes. Also, it is worthwhile to follow the example of Cvitanić

[37] Note that in our model, the time horizon is infinite: $t$ is not limited by some upper boundary $T < \infty$, as opposed to the model in Cvitanić et al. (2018).

[38] For the definition of $\sigma$-algebra, see, for example Tolonen (2017), the earlier and Master's Thesis by the author. The thesis is unfortunately written in Finnish, and there we study the properties of backward stochastic differential equations. For a more coherent or an English source, you can find solace in Durrett (2010).

[39] For the definition of Wiener measure, see Durrett (2010), or any introduction level measure or probability theory textbook.

[40] We say that a random variable $\tau$ is a stopping time in some filtrated probability space, if $\{\tau \leq t\}$ for all $\tau \in [0, \infty]$.

[41] See, as pointed out in Cvitanić et al. (2018), the theorem 1.1.8. in Stroock et al. (2006).

[42] Brownian motion is often described as an *almost surely* continuous Wiener Process $W_t$ with a starting point $W_0 = 0$ with independent increments and stationary increments spanned by a normal distribution – that is, the output has independent and stationary increments by definition. By the phrase "almost surely" we denote events which happen with a probability 1 with regards to some probability measure. In our case, the probability measure is $\mathbb{P}$: in such case we denote $\mathbb{P}$ -almost surely.

[43] By a Polish space, we mean a separable and topologically complete space. For broader definition or an overall lesson in topology, see, for example Kuratowski (1966).

[44] For further notes in measure theory, see any exhaustive analysis textbook covering the Lebesgue measure, or for short introduction, Durrett (2010) is feasible.

et al. (2018) and note that the processes $\phi$ are so-called *non-anticipative*, i.e. that for all $t$ and $x \in \Omega$, $\phi(t, x_t) = \phi(t, x_{\min\{\cdot,t\}})$.

Furthermore, we need to construct martingales. We call a probability $\mathbb{P} \in$Prob$(\Omega)$ a *semi-martingale*, if the corresponding output process $X_t(x)$ is a semi-martingale[45] under $\mathbb{P}$. Under some mildly technical details[46], for any $t \in [0, \infty)$, we can also define the quadratic variation $\langle X \rangle_t$ for $X_t$ $\mathbb{P}$-almost surely. Furthermore, we assume that the other set-theoretic axioms and hypotheses described in Cvitanić et al. (2018) section 2.1. hold, in order to avoid problems with the continuum when relaxing the control space.

Now, let us define the controlled state equation – that is, the equation according to which the total output $X_t$ evolves. The control process – the agent's effort – $\alpha$ is an $\mathbb{F}$-optional process, which attains in $A$, which is a subset of some finite-dimensional Borel-set. The requirement of $\mathbb{F}$-optionality in our context is well-defined, as pointed out in Cvitanić et al. (2018). Given a starting point or initial data $X_0$, the controlled state equation for this problem is defined as a stochastic differential equation spanned by a $n$-dimensional Brownian motion $Z_r$,

$$X_t = X_0 + \int_0^t \lambda_s(X, \alpha_s)\, \mathrm{d}s + \int_0^t \sigma\, \mathrm{d}Z_s, \tag{3.4}$$

where $\sigma \in \mathbb{R}$ is a constant, $\alpha$ is the agent's effort[47], $t \in [0, \infty)$ and $\lambda_r$ is a *controlled coefficient*

$$\lambda : \mathbb{R}_+ \times \Omega \times A \to \mathbb{R}^n, \tag{3.5}$$

$\mathbb{F}$-optional for all $\alpha \in A$.

We call a weak solution of the controlled state equation (3.4) an admissible control model, which we define as a pair $\mathbb{M} := (\mathbb{P}, \alpha) \in \text{Prob}(\Omega) \times A$[48], for which there is a $n$-dimensional $\mathbb{P}$-Brownian Motion $Z^{\mathbb{M}}$, such that

$$X_t = X_0 + \int_0^t \lambda_s(X, \alpha_s)\, \mathrm{d}s + \int_0^t \sigma\, \mathrm{d}Z_s^{\mathbb{M}} \tag{3.6}$$

$\mathbb{P}$-almost surely. Furthermore, we denote the collection of all such admissible control

[45]We define semi-martingales as a standard sum of a local martingale and some adapted process with finite variation. For technical details and more information, see for example Durrett (2010).

[46]For the technical details in question, see Karandikar (1994).

[47]Although we act as it was an elementary task, quantifying the agent's effort, or quantifying any abstract object from the real world, is in fact no trivial task. This problem can be reduced to the famous discussion in Hegel (1832, 2010), where it is argued that the argument of whether a *Being*, or effort, could quantified, since it can either increase or decrease, is profound and meaningless, since numbers are "never entirely present [in the real world]" (Carlson 2002, pp. 2028). Without going into too much details, as explained in Carlson (2002), Hegel overcomes this burden by dividing *Beings* into Quality, Quantity and Measure: we find the agent's effort – along many other *quantities* within the world of principal-agent problems – within the intersections of these three divisions.

[48]We also require that for the preimage of the data $X_0$ it holds that $\mathbb{P}(X_0^{-1}) = \delta_{\{X_0\}}$, where by $\delta$ we denote the Dirac delta function. In addition, for the details regarding the definition of $Z^{\mathbb{M}}$, see Cvitanić et al. (2018).

models as $\mathcal{M}$ – in other words, for all admissible models, it holds that $\mathbb{M} \in \mathcal{M}$ for all $\mathbb{M}$[49]. As in Cvitanić et al. (2018), we assume that for any $\sigma$ it holds that

$$\mathcal{M} \neq \emptyset. \tag{3.7}$$

Furthermore, as implied in Cvitanić et al. (2018), according to the Girsanov Theorem[50], all weak solutions spanned by any $\alpha$ and $\alpha'$, $\alpha \neq \alpha'$, are equivalent. To finish and for future use, let us introduce sets

$$\mathcal{P}(\alpha) := \{\mathbb{P} \in \operatorname{Prob}(\Omega), (\mathbb{P}, \alpha) \in \mathcal{M}\}, \quad \mathcal{P} := \bigcup_{\alpha \in A} \mathcal{P}(\alpha) \tag{3.8}$$

and

$$\mathcal{U}(\mathbb{P}) := \{\alpha \in A, (\mathbb{P}, \alpha) \in \mathcal{M}\}, \quad \mathcal{U} := \bigcup_{\mathbb{P} \in \operatorname{Prob}(\Omega)} \mathcal{U}(\mathbb{P}). \tag{3.9}$$

Finally, we are done with the very short introduction of our mathematical model[51], and we can move on to formulate the problems of the principal and the agent.

## 3.2 Agent's Problem

In this section, we introduce crucial concepts from the agent's perspective, and furthermore define his objective function.

First, in order to fit the problem to our mathematical model, we need to introduce a cost of effort function in order to reflect the agents' feelings of burden when he exerts effort through his actions. Let

$$h : \mathbb{R}_+ \times \Omega \times A \to \mathbb{R}_+ \tag{3.10}$$

be a cost function[52], which is a measurable function, and $h(x, \alpha) =: h(\alpha)$ be $\mathbb{F}$-optional for all $\alpha \in A$. Furthermore, let us assume that the cost function $h(\alpha)$ is continuous, increasing and convex. To make matters easier, let us normalize the reservation utility to 0, and furthermore

[49]For simplicity, we assume that all further control models $\mathbb{M}$ are admissible.

[50]For the Girsanov Theorem, see for example the wonderful and almost limitless resource Øksendal (2003). Furthermore, as *again* explained well in Cvitanić et al. (2018), the Girsanov Theorem also implies that any control model $\mathbb{M} \in \mathcal{M}$ spans a weak solution (3.6) for a driftless version of our control model.

[51]For more technical readers, Cvitanić et al. (2018) offers an extremely in-depth introduction to the mathematical side of principal-agent problems. We deliberately left out some of the details in order to keep the section lean, especially regarding the limits of the quadratic variation and the volatilities of $X_t$: this is justified, since we are not aiming to replicate the fascinating proof of the theorems presented in Cvitanić et al. (2018), but to merely apply the results for our own purposes.

[52]Do note that in Cvitanić et al. (2018), the cost function of effort is denoted $c$. In order to not mix-up the cost function between the compensation, which is denoted $C$ in Sannikov (2008), we decided to use notation $h$.

that $h(0) = 0$ – the agent only accepts contracts which result in positive expected utility. Furthermore, let us assume that there exists some $\gamma_0 > 0$, such that $h(\alpha) \geq \gamma_0 \alpha$ for all $\alpha \in A$.

Now, we recall that the process $X$ is called the *output process*, and as before, we call $\alpha$ the agent's *action* or *effort*. The agent exerts effort by choosing $\alpha$ in the controlled state equation (3.4) and controls the process[53], and is subject to the cost of effort $h(X, \alpha)$. We assume that the output process $X$ is observable by both principal and the agent – only the agent knows his effort $\alpha$. Furthermore, in this game, the duty of the principal is to use her observations of $X$ and incentivise the agent to exert effort tied to his cost function.

Before the agent starts working for the principal, the principal offers him a contract. In the contract, the principal promises the agent a non-negative, continuous flow of compensation, based on the principal's observation of the output:

$$C_t(X_s), \tag{3.11}$$

which can attain values in $[0, \infty)$, and where $0 \leq s \leq t$. Since the principal does not observe the effort of the agent, the compensation $C_t$ is $\mathcal{F}_T$-measurable. Additionally, we introduce a general utility function $U_A : \mathbb{R} \to \mathbb{R}$ for the agent. We assume that this utility function is invertible and bounded.

Furthermore, in order to ensure that the upcoming objective function is well-defined across the infinite horizon[54], we make two sets of assumptions: first, to allow different types of utility functions instead of only the general $U_A$, we assume that the compensation flow $C_t$ and the control variable $\alpha$ have sufficient exponential moments. Secondly, we want that the cost function $h$ and the utility with regards to the compensation flow $U_A(C_t)$ belong to some bounded norm space – that is, for some $p > 1$, the compensation-based utility and the cost function are integrable:

$$\sup_{\mathbb{M}} E^{\mathbb{P}} \left[ \int_0^\infty h_t(X, \alpha)^p \, \mathrm{d}t \right] < \infty, \tag{3.12}$$

,

$$\sup_{\mathbb{P}} E^{\mathbb{P}} \left[ |C_t(X)|^p \right] < \infty, \tag{3.13}$$

and

$$\sup_{\mathbb{P}} E^{\mathbb{P}} \left[ |U_A(C_t(X))|^p \right] < \infty. \tag{3.14}$$

We call compensation flows satisfying the criteria (3.13) and (3.14) *contracts*, and we write $C_t \in \mathcal{C}$, where $\mathcal{C}$ is the collection of all contracts.

[53]Or, to further specify a mention in Cvitanić et al. (2018), the agent controls the distribution of each of the time-points in the output process.

[54]For the cost function, the assumption of convexity would have sufficed.

The final piece missing from the agent's objective function is the discount rate: as in Sannikov (2008), we assume that the discount rate for the flow of profit is a constant[55] $r \in \mathbb{R}$.

We now introduce the agent's objective function as a dynamic value function. If the agent chooses effort level $\alpha_t$, $0 \leq t < \infty$, his objective function is

$$J_A(t, \mathbb{M}) := E^{\mathbb{P}}\left[ r \int_0^\infty e^{-rt}(U_A(C_t) - h(\alpha_t))\,\mathrm{d}t \right] \tag{3.15}$$

for all $\mathbb{M} = (\mathbb{P}, \alpha) \in \mathcal{M}$, at a time $t$. Now, our assumptions (3.12) and (3.14) ensure that the objective function is always well-defined. To finish defining the agent's problem, we follow the notation used in Cvitanić et al. (2018) and define the collection of optimal control models. We first reiterate that the agent's goal is to optimally choose his effort, given the compensation of the principal – that is, the agent maximises his *value function* $V_t$[56] which we denote as

$$V_t := V_A(t) := \sup_{\mathbb{M}} J_A(t, \mathbb{M}). \tag{3.16}$$

Furthermore, if $V_t = J_A(t, \mathbb{M})$, then we denote such model by $\mathbb{M}^*$ – an optimal response to contract $C_t$. We denote the collection of all such optimal control models by $\mathcal{M}^*$. Before moving on to the principal's side of the story, we point out that as opposed to the static value function in Cvitanić et al. (2018), in our model the value function $V_A$ is the agent's dynamic value or *continuation value*, where the initial time is $t$ – the point at which the principal offers the contract. As pointed in Cvitanić et al. (2018), this dynamic property plays a crucial role in characterising the problem of the principal, which we are to construct in the following section.

## 3.3 Principal's Problem

As explained in the previous sections, the principal-agent problem is a problem of two parties. In the previous section, we introduced the first player, the agent, and defined agent-related notation, together with agent-related assumptions. In the end, the agent's job is to match his effort to the compensation provided by the principal, who in turn receives profit when the agent exerts effort.

As noted in Cvitanić et al. (2018), we restrict the menu of contracts offered by the principal that are optimal for the agent – that is, we consider only contracts $C_t \in \mathcal{C}$ for which $\mathcal{M}^* \neq \emptyset$ – this is feasible, since the agent knows that the principal can offer an optimal contract for the

[55]The model could be expanded by adding, instead of a constant rate $r$, a random flow of discount rate $r_t^{\mathbb{P}}$. Discount rate with a random flow would reflect the properties of discount rate changing in time: the constant one is assumed to reflect the knowledge of the discount rate at the time of the contract: in addition, this assumption is true to the benchmark setting of Sannikov (2008), and it leaves room for further exploration.

[56]Note that we shorthand the notation and drop the $A$ from the subscript to make the notation more readable in characterising the optimal contract.

agent. Furthermore, when we define the agent's problem, we normalize his reservation utility to 0 – that is, the menu of contracts provided by the principal is further restricted, and finally we have a feasible contract menu

$$\Xi := \{C_t \in \mathcal{C}, \mathcal{M}^* \neq \emptyset, V_t(C_t) \geq 0\}, \tag{3.17}$$

or to open up the notation, the principal offers the agent bounded flow of compensation $C_t \in \mathcal{C}$, which makes the agent play an optimal response $\mathbb{M}^*$ to the compensation, subject to the constraint

$$\sup_{\mathbb{M}} E^{\mathbb{P}} \left[ r \int_0^\infty e^{-rt} (U_A(C_t) - h(\alpha_t)) \, \mathrm{d}t \right] = V_t(C_t) \geq 0. \tag{3.18}$$

In addition, the principal of course maximises her profit while offering the contract from menu $\Xi$. As with the agent, let us introduce a non-decreasing and concave[57] utility function $U_P : \mathbb{R} \to \mathbb{R}$ for the principal. Now, the objective function of the principal is

$$J_P(t, C_t) := E^{\mathbb{P}} \left[ r \int_0^\infty e^{-rt} U_P(l_t - C_t) \, \mathrm{d}t \right], \tag{3.19}$$

where the process $l_t$ is called the *liquidation function*, with which the output process is transformed into monetary value for the principal. The principal's goal is to optimally choose the compensation based on her information at a time $t$:

$$V_{P,t} := V_P(t, C_t) := \sup_{\{C_t \in \Xi\}} J_P(C_t). \tag{3.20}$$

Furthermore, we state that we are only interested in contracts which provide non-negative profit for the principal, $V_{P,t} \geq 0$, since we assume that the principal is the one offering the contract and has – generally speaking – no incentives to cause negative profit for herself.

We have now introduced our benchmark setting for the principal-agent problem at hand: recall, that the goal was to formulate the benchmark problem as a continuous-compensation, infinite-horizon principal-agent problem first built in Sannikov (2008), and built the mathematical notation and the level of description to prepare for turning this into a dynamic programming problem of Cvitanić et al. (2018). First, in section 3.1 we formulated our problem in to the world of continuous random variables[58] and build the foundation to the control set with the controlled state equation (3.4). Then, we used this nicely built foundation to formulate our problem of the incentive-loving agent. Finally, when we had the mathematical foundation and the reactive player, the agent, we combined these to form the problem for the principal, whose

[57] Do note that we do not impose a strictly concave utility function to allow risk-neutrality for the principal.

[58] Again, we skip some of the details covering stochastic processes, and instead refer to Cvitanić et al. (2018) and Cvitanić et al. (2016).

responsibility is to offer the contract in the first place, and thus, if you allow, whose responsibility is to make the first move in our strategic game. In the following section, we finish setting up the benchmark problem and examine the possibility of retiring the agent

## 3.4 Chance for Retirement – Finalising the Setting

In this section, we examine the possibility to retire the agent, and discuss the implications of the retirement to the optimal contract.

Before venturing deeper into the dynamic optimisation and discussing the game theoretic properties of our problem, we impose an additional assumption: the principal can retire the agent. Based on the agent's continuation value $V_t$, the principal must specify the flow of the agent's consumption flow $c_t$. As in Sannikov (2008), the notion of the principle retiring the agent with value $U_A(c_t)$ can be introduced by setting a utility $U_A \in [0, U_A(\infty))$, where $U_A(\infty) := \lim_{c_t \to \infty} U_A(c_t)$: to retire the agent, the principal offers the agent a constant consumption $c$ with zero effort. We denote the profit of the principal when retiring the agent as

$$V_{p,0}(U_A(c_t)) = -c. \tag{3.21}$$

We call $F_0$ the principal's *retirement profit*, as notated in Sannikov (2008). If the optimal contract exists, the retirement mechanism deploys a limitation to the optimal contract. Citing Sannikov (2008), outside the retirement, the first and obvious limitation is that the optimal contract must be characterised at some point $V_p \geq V_{p,0}$. Secondly, Sannikov (2008) characterises a so-called *smooth-pasting condition*[59]: for some point of the agent's continuation value $V_{t,gp} \geq 0$, the smooth-pasting condition is satisfied if $V_p'(V_{t,gp}) = V_{p,0}'(V_{t,gp})$[60]. Thirdly, the retirement occurs at a *retirement time* $t^*$, when $V_t$ hits 0, the agent's reservation value, or $V_{t,gp}$ for the first time[61].

## 3.5 Comments on the Problem

Before delving deeper into our game, let us use a moment to discuss this set-up. We are going to briefly discuss the properties of the continuation value $V_t$ and examine the game theoretic properties of the problem. Then, we go over the major assumptions used in the

[59]The smooth-pasting condition ensures that the Principal's value function is indeed maximised with the choice of $V_t$. The condition in the context of canonical diffusion processes is neatly explored in, for example, Manuelli (2007).

[60]Where we, for the sake of convenience, denote the partial derivative with regards to time with the primes.

[61]The mechanisms leading to retirement are more carefully explained in the appendices of Sannikov (2008) – we also explain the mechanics regarding our optimal contract in the following chapter after we have characterised our optimal contract.

formulation of the problem, and then discuss the used assumptions against earlier literature: mainly together with the articles Holmström et al. (1987), Sannikov (2008) and Cvitanić et al. (2018). This is fruitful, since some of the assumptions are of technical nature, while the others are to fundamentally differentiate the problem against empirical observations in the literature.

The *continuation value* of the agent $V_t$ is the total utility that the principal expects the agent to accumulate in the future, given any moment of time $t$. This continuation value plays a crucial role in the characterisation of the optimal contract. As motivated in Sannikov (2008), the continuation value plays determines what is the amount of compensation the agent receives, what is the effort level he is supposed to choose, and how the value $V_t$ itself changes in time.

To characterise the optimal contract, in her design the principal accounts the agent's continuation value $V_t$ at a time $t$ to compute the best reaction of the agent, and then based on agent's feedback she characterises the optimal contract to maximise her profit: a so-called *Stackelberg differential game*[62]. In the game, the goal of the principal is to maximise her value function. She maximises the value function by offering the agent a compensation flow for which the agent will react optimally. Based on the agent's continuation value $V_t$, the principal must specify the flow of the agent's consumption flow $c_t := c(V_t)$[63], the recommended effort level $\alpha_t := \alpha_t(V_t)$ and some rules, according to which the agent's continuation value $V_t$ moves in time. As in all principal-agent problems, the recommended effort level must be accepted and agreed upon by the agent, and the characterised contract must maximize the profit of the principal.

To reiterate, we have a problem with an agent and a principal. The agent exerts effort to produce output $X_t$, and the principal incentivises him with a continuous stream of compensation $C_t$. We then make assumptions crucial to the set-up:

- The output process evolves according to the controlled state equation (3.4),
- principal can not observe the agent's effort, only the output process $X_t$, and that
- the principal can choose not to employ the agent and that the agent's reservation utility can be normalized to 0.

We also made a further assumption that for all volatilities $\sigma$, we can always find a weak solution for the model. In other words:

- for all $\sigma \in \mathbb{R}$, $\mathcal{M} \neq \emptyset$ holds.

[62]The Stackelberg game approach in the context of principal-agent problems is explored for example in Long et al. (2010).

[63]Note the difference between the two consumption flows $C_t$ and $c_t$: $c_t$ is the a priori consumption flow which the principal has yet to specify according to the agent's actions, while $C_t$ reflects the agent's actions in the principal's maximisation problem: the compensation flow $C_t$ reflects the compensation based on the principal's view on the agent's future expected value.

Moving on, we defined a cost function $h$, for which we made two distinct assumptions, which both describe the nature of the cost function:

- that $h$ is continuous, increasing and convex (in $\alpha$), and that
- there exists some $\gamma_0 > 0$, such that $h(\alpha) \geq \gamma_0 \alpha$ for all $\alpha \in A$.

Then we made up a discount rate $r$ and utility functions $U_A$ and $U_P$ for both of our players, regarding which we made several assumptions to ensure that the agent's objective function is well-defined:

- $U_A$ is invertible and bounded, and $U_P$ is non-decreasing and concave,
- $C_t$ and $\alpha$ need to have sufficient exponential moments with regards to possible exponential $U_A$,
- $h_t$ is integrable and $U_A$ is bounded as in equations (3.12) and (3.14), and that
- the discount rate $r$ is constant.

Finally, the problem is fully characterised with allowing the agent to retire. With this we mean that the principal has a chance to retire the agent and offer him a constant flow of compensation, after which the agent exerts no effort.

Note that some of the technical "choices" along setting up the problem could be listed as assumptions: for example, instead of *announcing* that further on we only deal with contracts satisfying the equation (3.12), we could *assume* that all the principal only offered contracts satisfying the condition[64].

Certainly, the key defining assumption in our model is that the compensation is paid continuously, and not as a lump-sum in the end of the period. Seemingly, this characteristic

Comparing these assumptions to earlier literature, one of the major contributions compared to the problems of Holmström et al. (1987) and Sannikov (2008), is the assumptions regarding the utility functions. Instead of exponential utilities, we only assume $U_A$ to be invertible and bounded, and $U_P$ to be non-decreasing and concave. The assumptions of concavity and boundedness reflect the earlier literature: as reviewed in Friend et al. (1975), the these types of functions do not assess for the income effect. In addition, these assumptions are required in order to ensure that the expectation (3.14) exists, which in turn is required for the optimal contracts to exist. The assumption for invertibleness is required for the optimal contract to exist, and the principal's assumption of non-decreasing mirrors the empirical qualities of principals: the principal gains more utility on larger profits[65].

[64] This might feel like beside the point, but we liked to add that despite only those items above are listed as assumptions, there are several more hidden – perhaps unintentionally – choices that we made along the way while setting up the principal-agent problem.

[65] As a side note, this assumption is not entirely trivial, but will suffice for the large majority of microeconomic problems.

It is also noted that in Holmström et al. (1987), the cost function for the agent is monetary: they argue that the monetary cost function reflects an opportunity cost for not participating in some other income generating activity. In our model, the cost function is separated from the monetary realm, which is reflected by the utility function $U_A$. Instead, the cost of exerting effort is characterised as a separate function $h$, which mirrors the quantified burden of the agent – not necessarily a monetary cost.

Now that we have a clear view of our model, its assumptions and the key differences regarding to the literature, we can move on to the reduction of the problem, and characterising the reduced problem.

# 4. Reduction of the Problem and Finding the Optimal Contract

This chapter provides the main results of the thesis: the restriction of the menu of contracts and characterising the optimal solution in the continuous-compensation, infinite-horizon setting. The sections 4.1 and 4.2 include the main technical contributions of the thesis.

We first follow Cvitanić et al. (2018) on the reduction of the problem to a dynamic programming problem: we specify the game from the control perspective of the principal, introduce the relevant Hamiltonian functionals and use them to describe the family of restricted contracts. Furthermore, we show that the restricted family of contracts indeed maximise the principal's problem, and finally we examine that under the retirement option, the optimal contract still exists.

## 4.1 Restricting the Menu of Contracts

Let us start by defining the continuation value of the agent and the chance of retirement to our problem.

### 4.1.1 Finding the Hamiltonian

Given the consumption $c_t$ and the recommended effort level $\alpha_t$, the change in agent's continuation value can be written as[66]

$$\mathrm{d}V_t = r(V_t - U_A(c_t(V_t)) + h(\alpha_t(V_t)))\,\mathrm{d}t + rY(V_t)(\,\mathrm{d}X_t - \alpha_t(V_t)\,\mathrm{d}t), \tag{4.1}$$

where $rY(V_t)$ is the sensitivity of the agent's continuation value with regards to his output. For the last term we notice that

$$\mathrm{d}X_t - \alpha_t\,\mathrm{d}t = \sigma\,\mathrm{d}Z_t^{\mathbb{M}}, \tag{4.2}$$

[66]Please note that this is the exact equation provided in equation (3) of Sannikov (2008). In addition and just to clarify for the reader, this is the exact point where we differ from Sannikov's script and start characterising the Hamiltonian functional for the problem, as per Cvitanić et al. (2018). The precise derivation of this difference equation is also further explained in Sannikov (2008).

and thus change in the continuation value can be written as

$$\mathrm{d}V_t = r(V_t - U_A(c_t) + h(\alpha_t))\,\mathrm{d}t + rY(V_t)\sigma\,\mathrm{d}Z_t^{\mathbb{M}}. \tag{4.3}$$

We introduced the agent's sensitivity $rY(V_t)$ in order to reflect how the agent's valuation of exerting effort changes when his incentives change: if the agent deviates from his current level of effort, the change is only the drift of $X_t$. Thus, the agent has incentives to choose effort which maximises the expected change in $V_t$, when taking the cost of effort into account: in other words, $r(Y(V_t)\alpha_t - h(\alpha_t))$. As noted in Sannikov (2008), exposing the agent to risk is costly, the optimal contract is spanned at a lowest level of effort in the family of optimal effort levels $\alpha_t$. We denote this level as $\gamma(\alpha_t)$, which is increasing in $\alpha_t$, and specifically:

$$\gamma := \gamma(\alpha_t) = \min\{Y(V_t) \in [0,\infty) : \alpha_t \in \arg\max\{Y(V_t)\alpha_t - h(\alpha_t)\}\}. \tag{4.4}$$

Thus, the requirement in finding the optimal contract is that the optimal contract is spanned by a value function $V_t$, which evolves according to

$$\mathrm{d}V_t = r(V_t - U_A(c_t) + h(\alpha_t))\,\mathrm{d}t + r\gamma\sigma\,\mathrm{d}Z_t^{\mathbb{M}}. \tag{4.5}$$

We have now characterised the difference equation of the agent's continuation value. Our next step is to construct a *Hamiltonian functional* $H$, which describes the dynamics of the process. The Hamiltonian is the key in solving our dynamic optimisation process[67]. We use a so-called *Hamilton-Jacobi-Bellman (HJB) Equation*[68] to connect the Hamiltonian $H$ to the agent's continuation value function:

$$\partial_t V_A(t)\,\mathrm{d}t + H = 0, \tag{4.6}$$

where by $\partial_t$ we denote the partial derivative of the value function with regards to the time $t$.

We are now posed with a familiar question: in order to find the Hamiltonian $H$, we need the partial derivative of our value function. As the value functions presented in chapter 3.2 present the *á priori* state of the game, we need to find another way to solve for the partial derivative.

We find solace in our mathematical foundation: the value functions are spanned by $X_t$, which in turn can be characterised as *Itô Diffusion Processes*[69]. When we combine this result

[67] More in-depth introduction and background for the functional in dynamic programming problems can be found, for example, in Fleming et al. (2006).

[68] Usage of HJB-equations is based on a so-called *Martingale Optimality Principle*, and is broadly expanded in, for example, Zhou et al. (1999).

[69] This result is easy to see as the control $\lambda_t$ is the mean for the process, a proof for this result is shown in the appendices of Sannikov (2008). For more information and for the exact definition of Itô processes, see, for example, Øksendal (2003).

with the fact that our value function $V_t$ is differentiable with regards to $t$ and $X_t$, we notice that for the value function, *the Itô Lemma*[70] holds:

$$\mathrm{d}V_A(t) = \partial_t V_A(t)\,\mathrm{d}t + W_t\,\mathrm{d}X_t + \frac{1}{2}\operatorname{Tr}\Big[\Gamma_t\,\mathrm{d}\langle X\rangle_t\Big], \tag{4.7}$$

where we denote $W_t := \partial_x V_A(t)$, $\Gamma_t := \partial_{xx} V_A(t)$, $\operatorname{Tr}\Big[\,\cdot\,\Big]$ as the *trace* of the matrix in the argument[71], and $\langle X\rangle_t$ as the *quadratic variation*[72] of $X_t$.

Carrying on, we combine the equations (4.6) and (4.7) to finally characterise the Hamiltonian functional as

$$H(t, X_t, V_t, W_t, \Gamma_t, \gamma) := \frac{1}{2}\operatorname{Tr}\Big[\Gamma_t\,\mathrm{d}\langle X\rangle_t\Big] + K(t, X_t, V_t, W_t, \gamma, \alpha_t), \tag{4.8}$$

where

$$K(t, X_t, V_t, W_t, \gamma) := \sup_{\alpha_t \in A} k(t, X_t, V_t, W_t, \gamma, \alpha_t), \tag{4.9}$$

and furthermore

$$k(t, X_t, V_t, W_t, \gamma, \alpha_t) := -r(V_t - U_A(c_t) + h(\alpha_t))\,\mathrm{d}t - r\gamma\sigma\,\mathrm{d}Z_t^{\mathbb{M}} + W_t\,\mathrm{d}X_t. \tag{4.10}$$

We are able to separate the trace of the matrix from the functional $K$ in equation (4.8), since the agent is not controlling the volatility, and thus the dependence on the process $\Gamma$ disappears. Furthermore, this allows us to focus on maximising the functional $K$, since the same control maximises the original hamiltonian $H$.

Utilising the controlled state equation (3.6)[73], we further reorganise the map $k$ in equation (4.10) into a form

$$k(t, X_t, V_t, W_t, \Gamma_t, \gamma, \alpha_t) = -r(V_t - U_A(c_t) + h(\alpha_t))\,\mathrm{d}t + W_t\lambda_t(X_t, \alpha_t)\,\mathrm{d}t + (W_t - r\gamma)\sigma\,\mathrm{d}Z_t^{\mathbb{M}}. \tag{4.11}$$

### 4.1.2 The Family of Restricted Contracts

Having defined the Hamiltonian functional, we follow Cvitanić et al. (2018) and begin the reduction of the menu of contracts. First, to ensure that the family of restricted contracts is well defined, we need to introduce the norms

$$||W||^p := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\left(\int_0^\infty |W_t|^2\,\mathrm{d}t\right)^{p/2}\right] \tag{4.12}$$

[70] Again, seek solace in Øksendal (2003). The presentation presented by us differs from the standard notation and the notation used by Cvitanić et al. (2018) since we assumed the volatility of the process $\sigma$ to be a constant.

[71] The functions $W_t$ and $\Gamma_t$ have an interesting role in the optimisation: the existence of $W_t$ is fairly straightforward to prove, and the existence of $\Gamma_t$ is a classic puzzle in the stochastic optimisation literature – see, for example Possamaï et al. (2015) – but for the scope of this thesis, we assume that these functions exists, and furthermore, that the trace $\operatorname{Tr}\big[\Gamma_t\,\mathrm{d}\langle X\rangle_t\big]$ is well-defined. These assumptions are heuristically justified on the so-called standard representation theorem: regarding the theorem and more discussion on these assumptions we refer to Nutz (2012) and Cvitanić et al. (2018).

[72] See, for example, Durrett (2010) or Øksendal (2003).

[73] Note that the controlled state equation can be equivalently written in a form $\mathrm{d}X_t = \lambda_t(X,t)\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t^{\mathbb{M}}$.

and

$$||Y||^p := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\sup_{0\leq t<\infty} |Y_t|^p\right] \tag{4.13}$$

for any processes $W_t$ defined as above and for any $\mathbb{F}$-optional, real-valued and right-continuous processes $Y_t$. Again, following Cvitanić et al. (2018) and utilising these norms, we now introduce a process $Y_t^W$, which is used to define the restricted menu of contracts.

**Definition 4.14.** *Collection $\mathcal{A}$ is the collection of all $\mathbb{F}$-predictable processes $W$ which satisfy the following conditions:*

*(i)* $||W||^p + ||Y||^p < \infty$ *for some* $p > 1$*;*

*(ii) given some initial value*[74] $v(0, X_0) \in C^{1,2}$,

$$Y_t^W = v(0, X_0) + \int_0^t W_s \,\mathrm{d}X_s - \int_0^t K(s, X_s, Y_s^W, W_s, \gamma) \,\mathrm{d}s \tag{4.15}$$

$\mathbb{P}$ *-almost surely for all* $\mathbb{P} \in \mathcal{P}$ *and for some* $t \in [0, \infty)$*; and*

*(iii) there exists a weak solution* $(\mathbb{P}^W, \alpha^W) \in \mathcal{M}$ *such that*

$$K(t, X_t, Y^W, W_t, \gamma) = k(t, X_t, Y^W, W_t, \gamma, \alpha_t^W) \tag{4.16}$$

$\mathrm{d}t \times \mathbb{P}^W$ *-almost everywhere*[75] *on* $[0, \infty) \times \Omega$.

In the definition 4.14, the condition $(i)$ ensures that the process $Y^W$ is well defined in the equation 4.15 of condition $(ii)$. Finally, the last condition $(iii)$ states that the argument which maximises the Hamiltonian (4.8) induces an admissible control model for the agent's problem: this ensures that the problem of an optimal control for the agent's problem is well defined. Furthermore, this is required when finding a solution for the principal's problem, as implied[76] by the feasible control menu (3.17).

In the following theorem, we state that when the principal offers a contract with continuous flow of compensation $C_t = Y_t^W$, the agent's value function coincides with the process $Y_t^W$, and the optimal reaction of the agent is to exert effort with the same process $\alpha_t$ which maximises the Hamiltonian $K$ in (4.8)[77].

[74] With the set $C^{1,2}$ we denote the set of once and twice continuously differentiable functions, in other words, if the function belongs to this set, its first and second derivatives exist and are continuous.

[75] With the notation almost everywhere, we denote that the solution exists on $[0, \infty) \times \Omega$, but we allow the possibility that some 0-measure subsets existed with regards to measure $\mathrm{d}t \times \mathbb{P}^W$. For more information, see any introduction level measure theory textbook.

[76] This deduction is further examined in Cvitanić et al. (2018).

[77] This proposition is directly applied from the proposition 3.3. in Cvitanić et al. (2018), and its purpose in this thesis is to check that the optimal control exists in our modified benchmark setting. As the reader sees, the theorem is fairly straight-forward to prove with constant volatility and discount factor, even in the infinite-horizon setting.

**Theorem 4.17.** *Let $v(0, X_0) \in \mathbb{R}$ and $W \in \mathcal{A}$. Then, the following statements hold:*

*(i)* $Y_t^W \in \mathcal{C}$*;*

*(ii)* $v(0, X_0) = V_t(Y_t^W)$ *and any pair* $(\mathbb{P}^W, \alpha_t^W)^{\in}\mathcal{M}^*$ *is optimal; and*

*(iii)* $(\mathbb{P}^*, \alpha_t^*) \in \mathcal{M}^*$ *if and only if* $K(t, X_t, Y_t, W_t, \gamma) = k(t, X_t, Y_t, W_t, \gamma, \alpha_t^*)\,\mathrm{d}t \times \mathbb{P}$ *-almost everywhere on* $[0, \infty) \times \Omega$*.*

*Proof.* (i) The proof of the first part is fairly straight-forward: from the definition (4.15) of $Y_t^W$, together with the property $(i)$ of the definition 4.14 and the constant nature of $r$ and $\sigma$, we see that $Y_t^W$ indeed satisfies properties (3.13) and (3.14) for some bounded and invertible utility function $U_A$.

(ii) For this proof[78], we utilise the definition of the agent's objective function, $J_A(t, Y_t^W)$, together with the definition (4.15) and the formulation (4.10), to characterise an equation

$$\begin{aligned} &J_A(t, Y_t^W) \\ =&v(0, X_0) - E^{\mathbb{P}}\Big[r \int_0^\infty e^{-rt}\Big(K(t, X_t, Y^W, W_t, \gamma) - k(t, X_t, Y^W, W_t, \gamma, \alpha_t^W)\Big)\,\mathrm{d}t\Big]. \end{aligned} \tag{4.18}$$

From the equation (4.18) we see that $J_A(t, Y_t^W) \leq v(0, X_0)$. Directly from this we see that $V_t(Y_t^W) \leq v(0, X_0)$. Now, these inequalitties together with the equation (4.18) and the Hamiltonian (4.8) concludes the proof.

(iii) From the sketch proof of $(ii)$ we notice that for all $(\mathbb{P}^*, \alpha_t^*) \in \mathcal{M}^*$ it must hold that

$$E^{\mathbb{P}}\Big[r \int_0^\infty e^{-rt}\Big(K(t, X_t, Y^W, W_t, \gamma) - k(t, X_t, Y^W, W_t, \gamma, \alpha_t^W)\Big)\,\mathrm{d}t\Big] = 0. \tag{4.19}$$

From the construction of (4.8), we see that this holds if and only if $\alpha_t^*$ maximises the hamiltonian $K(t, X_t, Y^W, W_t, \gamma)$ $\mathrm{d}t \times \mathbb{P}$ -almost everywhere on $[0, \infty) \times \Omega$, which concludes the proof.

□

Furthermore, and again following Cvitanić et al. (2018), we denote an action maximising the hamiltonian $K$ as $u^* \in \mathcal{U}^*$, where $\mathcal{U}^*$ is the collection of the maximising actions. To reiterate, the theorem 4.17 states that $\alpha_t^* = u_t^*$ is well defined. Furthermore, especially with its part $(ii)$, the theorem states that for all pairs $W \in \mathcal{A}$ and any $u^* \in \mathcal{U}^*$, the stochastic differential equation

$$X_t = X_0 + \int_0^t \lambda_s(X_s, \alpha_s^*)\,\mathrm{d}s + \int_0^s \sigma\,\mathrm{d}Z_s \tag{4.20}$$

has at least one weak solution, which is the pair $(\mathbb{P}^{*,W}, \alpha_t^{*,W})$.

[78]In truth, this is only a rough and heuristical sketch of the proof, mimicing the proof in Cvitanić et al. (2018).

## 4.2 Validating the Restricted Menu

Now that we have the restricted family of contracts, which span the agent's objective function with at least one weak solution, we can move on to the Stackelberg game's other side, which is the principal.

### 4.2.1 Principal's Side of the Story

In this section, we further define the principal's problem in the Stackelberg game, and furthermore show that the principal's value function is maximised even with the restricted menu of contracts.

The principal's solution can be identified as

$$V_{P,t} \geq \sup_{v(0,X_0)\geq 0} \underline{V}(v(0,X_0)), \tag{4.21}$$

where

$$\underline{V}(v(0,X_0)) := \sup_{W\in\mathcal{A}} \sup_{(\mathbb{P},\alpha_t^*)\in\mathcal{P}^*(Y_t^W)} E^{\mathbb{P}}\Big[r\int_0^\infty e^{-rt}U_P(l_t^* - Y_t^W)\Big]. \tag{4.22}$$

In order to characterise the optimal contract, we need to identify conditions under which the problem $\sup_{v(0,X_0)\geq 0} \underline{V}(v(0,X_0))$ indeed attains exactly the same value as the unrestricted value $V_{P,t}$. The lower bound $\sup_{v(0,X_0)\geq 0} \underline{V}(v(0,X_0))$ represents the principal's maximum value spanned by the random processes $Y_t^W$, while taking into account the reservation value[79] $v(0,X_0)\geq 0$.

Let us now show that in our benchmark setting, the lower bound indeed matches the unrestricted principal's value, and additionally characterise the solution.

**Theorem 4.23.** *Let $\mathcal{A}\neq\emptyset$. Then*

$$V_{P,t} = \sup_{v(0,X_0)\geq 0} \underline{V}(v(0,X_0)), \tag{4.24}$$

*and additionally any pair $(v(0,X_0)^*, W^*)$ which maximises the problem $\sup_{v(0,X_0)\geq 0} \underline{V}(v(0,X_0))$ spans the optimal contract $C_t^* = Y_t^{W^*}$ for the principal's problem.*

*Proof.* For this proof, we follow Cvitanić et al. (2018)[80]. In order to proof this theorem, we need to show that every contract – a continuous compensation flow – $C_t\in\Xi$ can be represented as $C_t = Y_t^W$ $\mathbb{P}$ -almost surely, and thus reduce our canonical principal's problem to solving the backward stochastic differential equation

$$v(0,X_0) = C_t - \int_0^t W_s \,\mathrm{d}X_s + \int_0^t K(s,X_s,Y_s^W,W_s,\gamma)\,\mathrm{d}s, \tag{4.25}$$

[79]To reiterate, this condition ensures that the agent is willing to agree upon the contract.

[80]More closely, we follow the proof of their theorem 4.2, which is, extremely similar to our proof.

$\mathbb{P}$ -almost surely. It is easy to see that by definition,

$$\{Y_t^W | v(0, X_0) \geq 0 \text{ and } W \in \mathcal{A}\} \in \Xi. \tag{4.26}$$

Non-emptyness of $\Xi$ follows from the assumption $\mathcal{A} \neq \emptyset$. From the boundedness of $\lambda_t$, the property (3.13), and by Hölder's inequality[81], we see that for some $p > 1$, $C_t \in \mathbb{L}^p(\mathbb{P})$. That is, the contract belongs into some $\mathbb{L}^p(\mathbb{P})$-space[82]. As noted in Cvitanić et al. (2018), the boundedness of $\lambda_t$, together with the constant nature of $\sigma$ and $r$, we see that the functional $K$ is *uniformly Lipschitz-continuous*[83] in $(Y, W)$. Using the Hölder's inequality together with the property (3.12), we see that

$$E^{\mathbb{P}}\left[\int_0^\infty |K(t,0,0)|^{p''} \,\mathrm{d}t\right] = E^{\mathbb{P}}\left[\int_0^\infty \inf_{\alpha_t \in A} h_t(\alpha_t)^{p''}\right] \leq \sup_{(\mathbb{P},\mathcal{A})\in\mathcal{M}} E^{\mathbb{P}}\left[\int_0^\infty h_t(\alpha_t)^{p'}\right] < \infty, \tag{4.27}$$

for some $p' > p'' > 1$. We follow the example of Cvitanić et al. (2018), and state that according to Pardoux et al. (1990) and Briand et al. (2003), the existence and uniqueness of the representation (4.25) in $L^{p^*}([0,\Omega) \times \Omega)$ follows from the equation (4.27), where $p^* := \min\{p', p''\}$.

Finally, we see that the process $W$ satisfies the condition $(ii)$ of the definition 4.14. Furthermore, since $C_t = Y_t^W \in \Xi$, according to theorem 4.17, the process $W$ also satisfies the condition $(iii)$ in 4.14. The process also satisfies the condition $(i)$ by definition, and thus we have completed our proof: $W \in \mathcal{A}$. □

Theorem 4.23 is remarkable one, as it shows that the optimal contract is characterised from the reduced family of contracts. Cvitanić et al. (2018) even cited their version of the theorem to be one of the main contributions in their paper, and furthermore, they summarised the result to reduce a Stackelberg stochastic differential problem to a maximisation of a value function of a standard stochastic control problem. In this problem, the state variables are $(X_t, Y_t^W)$ with a control $W$.

### 4.2.2 Optimal Way to Retire

To further reiterate the solution to our continuous-compensation and infinite-horizon principal-agent problem, we combine the previous theorem 4.23 with the remark that the agent can retire:

[81] Let $p+q=1$. Now, in a $\mathbb{L}^p(\mathbb{P})$-space, for $\mathbb{P}$-measurable functions $f$ and $g$, we have $||fg||_1 \leq ||f||_p ||g||_q$. For notation and the definition of $\mathbb{L}^p$-spaces, see the following footnote. The book cited in the following footnote also contains more information on Hölder's inequality.

[82] This remark is useful due to the wonderful properties of $\mathbb{L}^p(\mathbb{P})$-spaces, notably the Hölder's inequality. In this context, we refer to a space in which the contracts $C_t$ have a property $||C_t||_p = (\int_0^\infty |C_t|^p \,\mathrm{d}\mathbb{P})^{1/p} < \infty$. For more information about these types of spaces, see, for example Adams et al. (2003).

[83] Lipschitz-continuity is a form of strong continuity in metric spaces. More specifically, a function $f$ is uniformly Lipschitz-continuous if, for all $x, y$ in the domain of $f$ and some constants $C \geq 0$ and $\alpha \geq 0$, we have $|f(x) - f(y)| \leq C\|x-y\|^\alpha$. This property is also sometimes known as Hölder-continuity, and for more information, see Adams et al. (2003).

we state the optimal contract, and show that the optimal contract exists with the boundary conditions set by the notion of retirement.

**Proposition 4.28.** *Let $\underline{V}(v(0,X_0)^*) \geq V_{P,0}(v(0,X_0)^*)$ be the unique solution to the principal's problem (4.21), where $V_{P,0}(v(0,X_0))$ is the value of the principal when retiring the agent. Let the so-called boundary conditions be*

$$V_P(0) = 0, V_{P,t}(V_{t,gp}) = V_{P,0}(V_{t,gp}), \text{ and } V_{P,t}^{'}(V_{t,gp}) = V_{P,0}^{'}(V_{t,gp}), \tag{4.29}$$

*which[84] characterise some point $V_{t,gp} \geq 0$. Now, if $v(0,X_0)^* \in [0, V_{t,gp}]$, the principal attains a value of $\underline{V}(v(0,X_0)^*)$, and the optimal contract is characterised as $C_t^* = Y_t^{W^*}$, where the process $Y_t^{W^*}$ evolves according to*

$$Y_t^{W^*} = v(0,X_0)^* + \int_0^t W_s^* \, \mathrm{d}X_s - \int_0^t K(s, X_s, Y_s^{W^*}, W_s^*, \gamma) \, \mathrm{d}s, \tag{4.30}$$

*which is spanned by the action $\alpha_t^*$ and the control $W^*$, and where $t \in (0,\infty)$, until the retirement time $t^* = \inf\{t | Y_t^{W^*} = 0, Y_t^{W^*} = V_{t,gp}\}$. After retirement, the agent attains a constant consumption flow of $-V_{P,0}(Y_{t^*}^{W^*})$, and exerts effort $\alpha_{t'} = 0$ for any $t^{'} \geq t^*$.*

*To reiterate, the optimal contract is defined as*

$$C_t^* = Y_t^{W^*} \text{ and } (\mathbb{P}, \alpha_t^*) \in \mathcal{P}^*(Y_t^W) \text{ for } t \in [0, t^*), \tag{4.31}$$

*and*

$$C_t^* = -V_{P,0}(Y_{t^*}^{W^*}) \text{ and } \alpha_t^* = 0 \text{ for } t \geq t^*. \tag{4.32}$$

*Proof.* [85] The limited solution is well-defined $\underline{V}(v(0,X_0)^*) \geq V_{P,0}(v(0,X_0)^*)$. In the lower bound, the optimal contract is characterised at the maximum value of 0 for the principal. Similarly to the proof of the theorem 4.23, $Y_t^{W^*} \in \Xi$, and since at any given time $t \in [0, t^*]$, the limitation $Y_t^{W^*} \in [0, V_{t,gp}]$ is a bounded subset of $R$, which guarantees the existence of the optimal contract. For the existence of the value function which satisfies the boundary conditions, we refer to the Lemma 3 in the appendices of Sannikov (2008). □

The proposition 4.28 ends our research: to reiterate, we established the problem of Sannikov (2008) and transferred its setting to a mathematically different problem. Then, we used the restriction of contracts approach of Cvitanić et al. (2018) to characterise the optimal contract: in doing this, we managed to transform the non-standard Stackelberg differential game into a standard stochastic optimisation problem, which is fully characterised at observation time $t$. At time $t$, the principal can restrict the menu of contracts offered and then compute the optimal continuous-compensation flow offered to the agent. In doing so the principal sacrifices

[84]Again, with the notation $V_{P,t}^{'}$ we refer to a partial derivative with regards time $t$.

[85]Note that this is not an exhaustive proof for the proposition, the reader should rather consider it as a sketch of a sort.

no value, even though the reduction allows her to know the reaction of the agent: as a reaction to the compensation offered, the agent will exert effort in optimal way – by this we mean that the reaction of the agent maximises the principal's value function. In addition, we allowed the possibility for the agent to retire. The chance of retirement does not jeopardise the optimal contract, but instead allows the agent to retire with a constant compensation flow.

This principal-agent problem and the corresponding benchmark setting have a room for extensions, and in addition, they interact with the previous principal-agent literature in an exquisite way. Let us discuss the implications of our research and conclude the thesis in the following, final chapter.

# 5. Discussion and Conclusion

In this chapter we discuss the microeconomic implications for the optimal contract in our setting. We characterise different utility functions to make the implications concrete, and furthermore reflect the results on previous literature.

## 5.1 On the Results

The results 4.23 and 4.28 prove to be quite fascinating: when the principal offers an admissible contract with continuous compensation $C_t^* = Y_t^{W^*}$, she knows that her offer spans a backwards deduction of a Stackelberg differential game. The principal knows, that the agent knows that the principal is offering an optimal contract for the agent, and thus the agent will choose the control $\alpha_t^* = u^*(t, X_t, Y_t^W, W, \alpha^*)$. This action maximises the objective function of principal over the admissible process $W$. To reiterate, instead of maximising the menu of all possible contracts $C_t$, the principal maximises over process $W$, for which she knows the agent's optimal action. In addition, theorem 4.23 ensures that by restricting the menu of possible contracts, she does not decrease her value function. This leads to an interesting dynamic: the continuous-compensation problem across continuous time is reduced to a single observation point $t$. At time $t$, the principal designs the optimal contract from the restricted menu, and thus spans an admissible and $\mathbb{F}$-predictable control process, according to which the agent exerts effort until retirement. However, there are few crucial remarks when viewing our results against the results in the establised literature.

### 5.1.1 Comparing the Results to Earlier Literature

In this section, we briefly discuss the results against the earlier literature. We focus on the results of Holmström et al. (1987), Sannikov (2008) and Cvitanić et al. (2018).

As noted in the literature review 2, one of the most important implications in Holmström et al. (1987) is that to construct an optimal contract, the principal requires high information about the agent's beliefs and preferences, in addition to the technologies the agent controls. This is an important remark, and the reduction of the control space tackles this dimension exceptionally: by reducing the number of contracts, the principal requires less information from the agent. The main result of Holmström et al. (1987) is that when assuming a single-period, constant effort output in a second-best scenario and when the agent's utility is strictly risk-averse and exponential, then the optimal contract is linear in the aggregate output. The linearity of our solution is of interest, and we return to it briefly.

The model of Sannikov (2008) does not take the possible savings of the agent into an account, and the same loophole is also extended to our benchmark setting. This could be further examined and discussed in a first-best setting, where the principal would have a knowledge of the agent's savings. However, in reality, this assumption is fairly unrealistic. In addition, it is to note that we did not explore the continuation paths of the optimal contract as in Sannikov (2008). In the continuation paths, the agent's dynamics could be better understood, especially in the realm of researching the wage fluctuations. Researching the continuation paths towards the retirement would be an important extension for our research. Furthermore, the authors discussed in Holmström et al. (1991) that the model of Holmström et al. (1987) is exceptionally well-suited for compensation over short time period: this reflects to both our result and the result of Sannikov (2008), for which the linearity is swayed in the long-horizon contracts. This was due to the retirement dynamics: in this sense, our optimal contract is exceptionally similar to the one in Sannikov (2008)[86].

The main result for Cvitanić et al. (2009) and Cvitanić et al. (2018) in the linearity sense is that the optimal contract for non-exponential preferences is a *non-linear function of aggregated output*, as opposed to the linear solution in Holmström et al. (1987). In the most general sense, our setting derives the same result. The optimal contract equation is heavily dominated by the Hamiltonian $K$. Furthermore, the Hamiltonian is of different when comparing to the Hamiltonian in Cvitanić et al. (2018), which is why the dynamics of the optimal contract differ, not to mention the characterisation differences between continuous and lump-sum payments. Regarding the linearity, however, with a specified selection of certain constants, our setting would also characterise a linear function of aggregate output, as in Holmström et al. (1987). Consider

$$\begin{aligned}
Y_t^W &= v(0, X_0) + \int_0^t W_s \,\mathrm{d}X_s - \int_0^t K(s, X_s, Y_s^W, W_s, \gamma)\,\mathrm{d}s \\
&= v(0, X_0) + \int_0^t W_s \,\mathrm{d}X_s \\
&- \int_0^t +r(V_t - U_A(c_t) + h(\alpha_t))\,\mathrm{d}t - W_t\lambda_t(X_t, \alpha_t)\,\mathrm{d}t - (W_t - r\gamma)\sigma\,\mathrm{d}Z_t^{\mathbb{M}}\,\mathrm{d}t.
\end{aligned}$$

With linear $\lambda_t$ and linear liquidation process $l_t$, the change in the value function would be linear in output.

[86]Do note, however, that we did not do rigorous analysis on the paths of the continuation value: however, the dynamics and the derived optimal contract are extremely similar.

## 5.2 Extensions to the Principal-Agent Literature

From the economic point of view, there are a plethora of interesting extensions to both our problem, and to the principal-agent problems in general.

As discussed in Cvitanić et al. (2018), the ability for the agent to control the volatility, either cost-less or with a cost, would be an interesting addition to our model. The ability to control volatility would reflect the agent's need to adjust the risk level in time, to reflect their different situations across time. The ability to control volatility would make the agent to be able to choose the level of risk-taking, perhaps in order to reflect the chance for higher value from the optimal contract. Previous literature is mainly revolving around processes, in which the agent can not control his volatility, and thus this extension would be a welcome addition in the upcoming research.

Additionally, when imposing the volatility control would increase the mathematical difficulty in finding a solution. For example, in principal-agent problems a numerical solution can be used to characterise an analytically challenging optimal contract. This side of the principal-agents was explored in Azar et al. (2018).

However, we are afraid that hasty extensions would be in naught: many of the extensions in the research of this area come with a cost of unnecessarily inflating the model, while adding only little economic interest. In our thesis, a wonderful addition would have been to calculate an optimal contract with a non-exponential, but bounded, utility function for the agent. However, the implications of this extension would have been unclear: as shown in Friend et al. (1975), exponential utilities, especially CARA utilities, reflect the empirical research on the risk-aversion among the agents. The literature on principal-agent problems have been revolving around the groundbreaking article of Holmström et al. (1987) for over 30 years: the classic article offers a simple mathematical foundation with intuitive and empirically verifiable results. However the literature is to be expanded, we can only hope that they come along with great empirical results.

# 6. Conclusion

In this thesis, we first charaterised a familiar setting, in which the agent receives continuous compensation, and in which the principal and the agent are forever tied to the contract. We model this game of contracting in a peculiar way: despite the continuous time with infinitesimal time between the payments and infinite time horizon, the characterisation of the optimal contract is tied to a single observation point $t$. At time $t$, the principal decides to discard all of the menus that the agent would not pick up, or for which the agent would react sub-optimally. By doing this, she creates a restricted menu of contract, for which she is able to identify the agent's optimal effort and optimal reaction. As a major result, we show that even in the infinite-horizon setting, this reduction does not decrease the principal's profits, and that the optimal contract can be characterised even when the agent has the option to retire. This optimal contract, in its most general form, is non-linear function in output. However, when imposing additional assumptions, the short-term linearity is achieved, and thus the solution reflects the results in earlier literature.

In addition, the insight that the optimal contract is characterised from a single observation point $t$ also ties the problem into the realm of approximating lump-sum principal-agent problems as continuous problems. This notion explains why the optimal contract of the continuous-compensation, infinite-horizon problem appears to have distinctly similar dynamics to the problem with finite time horizon and lump-sum payments, for example the problems discussed in Holmström et al. (1987) and Cvitanić et al. (2018).

Finally, the problem can be extended in obvious ways: with the technical contributions of this thesis, it would be natural to extend the model and allow the agent to choose the internal volatility of the output process, as well. This extension would mirror the economic property for the risk-loving agents, and furthermore motivate the research into exploring the optimal contracts for different types of agents. The notion of allowing the agent to control the volatility would also motivate the research of backwards stochastic differential equations, as the volatility control imposes additional technical problems for the model. Most importantly, possible extensions to this model would further increase the accumulate the knowledge on continuous-time principal-agent problems in the footsteps of Holmström et al. (1987), as well as moral hazard problems in general. Contracting and problems regarding hidden information will be relevant in the changing world, and preparing our economic frameworks to be as robust as possible will be paramount in the future of economics.

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